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THE TWO-NUCLEON INTERACTION AT LOW ENERGIES

by

© A. P. DEVIDAS MENON

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE TWO-NUCLEON INTERACTION AT LOW ENERGIES submitted by A. P. DEVIDAS MENON in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Physics.

Supervisor

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TO

MY LOVING PARENTS

ABSTRACT

In this work, the nucleon-nucleon interaction at low energies (below 300 MeV) is examined in a non-perturbation-theoretic manner. Two models are considered, the first being that where the interaction is mediated by scalar mesons (the scalar model), and the second, by pseudoscalar mesons (the pseudoscalar model).

The scalar model is treated first. Equations for the scattering and bound states are derived in configuration space. Explicitly non-local and energy-dependent potentials are obtained. In momentum space, an equation for the half-off-shell T-matrix is derived. The Born approximation for this model is then obtained. Half-off-shell T-matrix elements, and the phase shifts up to 230 MeV are calculated. For the bound state, a variational approach is taken to obtain a value of the coupling constant that gives the desired deuteron binding energy.

The pseudoscalar model is accorded a similar treatment. There are four scattering channels in this case, and particular attention is paid to the $S = 0$, $T = 1$ channel; it is treated analogously to the scattering state in scalar theory. The T-matrix is once again obtained. Finally, for the bound state, viz. the deuteron, configuration space potentials are obtained, and the equations for the s- and d-waves are derived.

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I INTRODUCTION

It had become apparent in the late 1920's, that nuclei could not contain electrons, because of their size, spin, statistics, etc. Following the discovery of the neutron in 1932, it was established that all nuclei were composed of neutrons and protons - known jointly as nucleons.

The next step, obviously, was to try to understand the nature of the forces that keep a nucleus intact. As the simplest, and in many respects, the most fundamental of all nuclei, is that consisting of two nucleons, such systems have undergone a great deal of surveillance and study, both experimentally and theoretically. A great deal of data on the two-nucleon system has been amassed, at various energies, and large quantities of theoretical work have accumulated in the literature. One reason for this is that it is hoped that perhaps understanding N-N forces would lead to further clarification of, and more tractable methods of handling, multi-nucleon systems, and also nuclear structure problems.

In this work, the 2-nucleon problem is taken up once again. The nucleons are non-relativistic, and hence the models will not be valid much higher than the single pion threshold (or about 300 MeV lab energy). In Section II, a very brief historical resumé is given of some of the many approaches taken. As this work involves a certain technique of quantisation, Section III is devoted to outlining it; the significance of some of the terms that

appear will also be pointed out. Section IV is concerned with the application of this technique to the first of two models, viz. the model with scalar meson exchange. Both momentum and configuration space representations are discussed, and equations for the scattering amplitude and the T-matrix are derived. Certain physical quantities are then extracted (within an approximation). A similar treatment is accorded the theory with pseudoscalar mesons, in Section V, and phase shifts are extracted, again within the limits of the technique of evaluation. Finally, the conclusions and a discussion of results are contained in Section VI.

The purpose of Appendices A - J is to remove the somewhat lengthy calculations (at least in some cases) from the main body of the text, thereby retaining some degree of continuity in it.

II HISTORICAL BACKGROUND

The problem of the interaction between two nucleons is one of the most fundamental in all of physics. Over the past four decades, it has been dealt with by a large number of physicists in a multitude of ways. Whereas, in 1960, the situation was such as to prompt the following quote of M.L. Goldberger[1]:

> "There are few problems in modern theoretical physics which have attracted more attention than that of trying to determine the fundamental interaction between two nucleons. It is also true that scarcely ever has the world of physics owed so little to so many. In general, in surveying the field, one is oppressed by the unbelievable confusion and conflict that exists",

the last fifteen years have to be considered as having been fairly productive.

There are a number of reasons why the N-N interaction warrants study; among them is the fact that for low energies, the N-N interaction is currently the most well-understood strong interaction process. Hence it has to be looked upon as a proving ground for ideas in elementary particle physics[2]. Also, an understanding of N-N interaction would facilitate the treatment of nuclear structure calculations to a great degree.

The purpose of this section is to outline, in brief, the history of such work. It has, of necessity, to be brief, because of the very large number of articles in the literature relevant to the problem. A fuller review would be very voluminous, and apart from the fact that such reviews are published from time to time, is not of prime importance here. The purpose of this section is to try and view the N-N interaction in its proper perspective, and to describe, albeit briefly, the major contributions toward solution of the problem, their relative merits, and their shortcomings and pitfalls, if any.

Basically, there have been two different approaches taken, viz. (a) phenomenological potential models, and (b) field-theoretical models.

II.1 Phenomenological Models

The prime motivation behind the use of potentials to describe what is, in fact, a strong interaction originally was the relative success of such an approach in electromagnetic and gravitational problems. It is phenomenological in that the potentials are constructed so as to reproduce existing data from experiments.

One of the earliest attempts was by Gammel, Christian and Thaler[3], and it was to fit all known data by a potential of the form

$$V(r) = V_C(r) + V_T(r)S_{12} \qquad \text{(II.1)}$$

The subscripts C and T stand, of course, for the central and tensor components of the potential, respectively, and S_{12} is the usual tensor operator

$$S_{12} = 3(\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

Eqn. (II.1) in fact represents the most general momentum independent potential which is local, conserves total angular momentum, parity and charge, is time-reversal invariant, and charge independent. However, a good fit was not possible, and in order to improve the situation, the restriction of momentum independence was removed; a term linearly dependent on the relative momentum was added (which leads to the Wigner[4] type of potential) to give

$$V(r) = V_C(r) + V_T(r)S_{12} + V_{LS}(r)\vec{L}\cdot\vec{S} \qquad \text{(II.2)}$$

Although with (II.2), fits were improved, they were still not satisfactory. In fact, if one assumes that the momentum dependence is purely linear, then only one potential V(r) can act in singlet-odd states, and another in all singlet-even states[4]. However, it can be shown that it is in fact impossible to describe <u>all</u> singlet-even states by a <u>single</u> central potential[5].

Subsequent to attempts of this nature, have come three of the most widely used potentials. In 1962, the Hamada-Johnston[6] and the Yale[7] potentials were con-

structed; neither was of the Wigner type. Both of these were vastly improved, in terms of data fits, over previous attempts; the price that was paid, was, of course, simplicity of form. The potentials are:

$$V_{HJ}(r) = V_C(r) + V_T(r)S_{12} + V_{LS}(r)\vec{L}\cdot\vec{S} + V_{LL}(r)L_{12}$$

with

$$L_{12} = [\delta_{LJ} + (\vec{\sigma}_1\cdot\vec{\sigma}_2)]L^2 - (\vec{L}\cdot\vec{S})^2$$

and

$$V_{Yale}(r) = V_C(r) + V_T(r)S_{12} + V_{LS}(r)\vec{L}\cdot\vec{S}$$

$$+ V_q(r)[(\vec{L}\cdot\vec{S})^2 + \vec{L}\cdot\vec{S} - L^2]$$

The Hamada-Johnston potential, for example, reproduced low energy (<315 MeV) 2N data better than any other potential to date; data such as effective range expansion parameters, deuteron parameters etc. $V_{HJ}(r)$ is a hard-core potential, with its functional form outside the core being a combination of Yukawa potentials.

The third potential which has met with reasonable success is the potential of Reid[5]. This, too, is non-local, in the sense of momentum dependence, and is of a different type in each angular momentum state. In addition,

both hard and soft cores can be made to fit the data; again, the radial dependence is of the Yukawa type. However, as with V_{HJ} and V_{Yale}, the Reid potential requires a fairly large number of parameters; this, of course, is an obvious consequence of attempts to fit different potentials in different angular momentum channels.

An even more recent development is that of de-Tourreil and Sprung[8]. Their potential is a "super-soft core" potential, and is of the following form:

$$V_{dTS} = \alpha_1 e^{-\beta r^4} + \left[\frac{\alpha_2 e^{-\beta r}}{r} + V_{OPEP}\right](1 - e^{-r^4})$$

Like the Hamada-Johnston and Reid potentials, this one also approaches the OPEP (one-pion exchange potential) tail at large r. The three differ from each other near the origin - hence the names "hard-core", "soft-core", and "super-soft core". However, the fit to data up to 300 MeV is fairly good for this potential as well.

Briefly, the current status of phenomenological potentials is the following: they all give (more or less) the same on-shell parameters such as phase shifts. And this is not sufficient to determine the validity of any particular potential. This is related to the fact, that apart from problems of parameter fitting etc., there is a more basic difficulty with regards to the phenomenological approach. Elastic processes are, in fact, insufficient to

determine, or test fully, an N-N potential[9]. Even if one has a theoretical two-nucleon potential which fits all phase shifts (which, in practice, one will never be able to obtain experimentally), one has to consider, in addition, off-shell phenomena such as nuclear bremsstrahlung. This is because, given a potential which gives a set of phase shifts for all angular momenta and energies, it is possible to construct a high order of infinity of potentials which will reproduce this data i.e. the class of phase equivalent potentials. Phenomenologically constructed potentials will therefore have little to say about off-shell phenomena. However, the current experimental work has not yet been able to conclusively test off-shell phenomena.

II.2 The Field-Theoretical Approach

The basic philosophy behind this approach (in fact, behind any field theory) is that the interaction of the system may be described by the exchange of particles. Due to the relationship between the mass of exchanged quanta, and the range of the force mediated by this exchange, it is possible, in practice, to describe the interaction at varying distances from the centre of the system, by the exchange of particles with the appropriate mass. A useful guideline in this respect has been the approach of Taketani, Nakamura and Sasaki[10]. The region of interaction is divided into three ranges: the classical region ($r \geq 2f$), the dynamical region ($2f \geq r \geq 1f$), and the core ($r \leq 1f$). In the classical region, it is expected that one pion exchange (OPE), by virtue of the pion mass, will dominate. When one gets into the intermediate (or dynamical) region, exchanges of other types of particle begin to enter; finally, in the core region, the situation is complicated enough to warrant the use of a phenomenological core.

The one-pion exchange potential (OPEP) tail is the only well-established meson-theoretic potential; in fact, its correctness has been established experimentally since as far back as the late fifties. Most, if not all, of recent work has gone into explaining the behaviour in the dynamical region, where one no longer has OPE dominance. Historically,

the success of OPEP motivated many attempts to derive a TPEP[11] (two-pion exchange potential). However, the successes of these initial attempts were limited, and their derivations were not entirely ambiguity-free.

In principle, to calculate the 2π exchange contribution, one needs all irreducible diagrams which have two pion lines joining the nucleons. Because of the infinite number of these, such a calculation is impossible. Therefore, in practice, one is forced to neglect certain diagrams. In fact, if one neglects radiative corrections, one is in fact neglecting what could possibly be sizeable contributions.

Some of the many papers published on TPEP are those of Sugawara and Okubo[12], Gammel et al.[13], and Partovi and Lomon[14,15]. When used with OPE models, these TPEP represent some improvement over OPEP; however, there are conceptual difficulties involved which make such calculations unreliable to a certain extent[2]. The basic difficulty, that of not being able to take into account all possible diagrams (or, equivalently, the problem with any perturbation theoretic approach) sets a limit on the accuracy of the models. In any case, TPEP generally suffers from problems in treating $N\bar{N}$ pairs and in treating the meson-meson interaction[17].

Chronologically, after the early TPEP attempts came the introduction of OBE[16] (one-boson exchange) models. These models assume that the NN force is meson-mediated, and that the exchanged systems may be adequately

represented by the meson resonances observed experimentally. This is based on the fact that in strong interactions, two or more particles as a group, tend to behave most of the time as a single particle with definite mass, width and intrinsic quantum numbers. The hope here, then, is that uncorrelated multipion contributions may be neglected, or at least represented by equivalent resonances. One advantage of OBEP models over TPEP models is that in the latter case, such contributions as 3π exchanges can never be included; another is that calculating single particle exchanges is a lot easier than calculating two-particle exchanges.

The type of heavy boson exchanged is restricted by the symmetries of the interaction e.g. they must have zero strangeness. One difference between the various OBE models is in the particular meson exchanged. Needless to say, the full list of heavy bosons known has not been exhausted by the various theories; by the same token, it should be mentioned that not all the particles that have been used, have in fact been experimentally observed. In addition, calculations are done assuming the bosons to be stable particles; however, this is not true. In short, OBEP suffers from uncertainties as to which bosons are "really there", and with regards to treating massive composite systems (e.g. ρ resonance) which decay rapidly, as single particles[17].

There are, of course, certain nice results from this approach, e.g. the prediction of a large spin-orbit force, and of a repulsive core (see ref. 2 for further references). However, these are only qualitative results. Among the OBEP can be mentioned those of Bryan, et al.[18], Green and collaborators[19], and the Japanese group[20]. An extensive list is tabulated in ref. 2, and a review of OBEP has also been done by Ogawa[21].

This, then, is a cursory glance at the situation; though brief, it shows that despite the great deal of work that has gone into the 2N problem, and despite the great strides that have been made, the problem is far from being solved. In this work, a fresh attempt is made to construct a field-theoretical model to describe the N-N system, free of divergences, and independent of perturbation theory. It will be seen that intrinsically non-local, and explicitly energy-dependent potentials are in fact necessary for a proper description of N-N phenomena. Although the equations involved are exceedingly complicated, approximations can be made to enable one to extract information from them.

III MATHEMATICAL FORMALISM

In this work, the interaction between the two nucleons is assumed to be carried via the exchange of mesons (scalar and pseudoscalar, in the two cases to be dealt with); this amounts to prescribing a field-theoretical model. The model is then quantised à la Capri[22]. In this section, the method of quantisation will be outlined. In essence, this will be a resumé of ref. 22.

In general, a relativistic quantised field ψ will consist of both creation and annihilation operators, due to the presence of both mass hyperboloids (or positive and negative energies) associated with the field. It is not just the relativistic nature of the field that is responsible for this - if one has a non-relativistic many-body problem, the same situation arises, because of excitations above the Fermi level, which leave holes behind. On the other hand, in a non-relativistic few-body problem, this is not so, and one can take $\psi(\psi^\dagger)$ to be a pure annihilation (creation) operator. It is precisely this feature of the non-relativistic problem that facilitates the quantisation procedure. In both the models to be dealt with, the nucleons are non-relativistic, and therefore the technique becomes directly applicable.

Consider the general class of equations of the type

$$(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M})\ \psi(t,x) = \int d^3x'dt'V(t-t';\vec{x}-\vec{x}')\psi(t,\vec{x})\psi^\dagger(t',\vec{x}')\psi(t',\vec{x}') \qquad \text{(III.1)}$$

This is, of course, a Schrödinger-type eqn. for the field ψ, which is both a non-local, and a non-linear equation. One first assumes that there exists a complete set of physical states (in the domain of $\psi,\psi^\dagger$), viz. the vacuum $|0\rangle$, one-particle states with definite momentum $|\vec{k}\rangle$, two-particle scattering states of definite momenta $|\vec{k}_1\ \vec{k}_2\rangle$, two-particle bound states labelled by the binding energy E_n and centre of mass momentum $\vec{p}$, $|E_n\ \vec{p}\rangle$, and so on. Furthermore, one assumes equal time commutation (or anti-commutation) relations, defined between pairs of physical states; for the vacuum and one-particle states, this yields

$$\langle 0|\,[\psi(t,\vec{x}),\psi^\dagger(t,\vec{y})]_\pm\,|0\rangle = \delta(\vec{x}-\vec{y}) \qquad \text{(III.2)}$$

$$\langle \vec{k}_1|\,[\psi(t,\vec{x}),\psi^\dagger(t,\vec{y})]_\pm\,|\vec{k}_2\rangle = \delta(\vec{k}_1-\vec{k}_2)\,\delta(\vec{x}-\vec{y}) \qquad \text{(III.3)}$$

The general formalism can be extended in a straightforward manner to n-particle states. But as one is primarily interested, in this case at least, in 2-nucleon systems, it is sufficient to go only as far as n = 2. The objects of interest are not the field ψ or the states $|\vec{k}_1..\vec{k}_n\rangle$, but rather the matrix elements of products of the field operators between these physical states; these matrix elements are then interpreted as the physical wavefunctions.

III.1 One Particle States

In order to obtain an equation for the two-particle wavefunction, it is necessary first to obtain the one-particle equation. The one-particle wavefunction is defined as follows:

$$u_k(t,\vec{x}) = \langle 0|\psi(t,\vec{x})|\vec{k}\rangle \qquad \text{(III.4)}$$

Quantisation amounts to taking the expectation value of eqn. (III.1) between an n-1 particle state, and an n-particle state, thereby yielding an equation for the n-particle wavefunction. In this case, n=1, and the one-particle equation is obtained by taking the expectation value of eqn. (III.1) between $|0\rangle$ and $|\vec{p}\rangle$. One then inserts complete sets of states between the field operators on the righthand side of (III.1). However, because $\psi(\psi^\dagger)$ is purely annihilation (creation), the only intermediate states that contribute will be one-particle states (between $\psi(t,\vec{x})$ and $\psi^\dagger(t',\vec{x}')$) and the vacuum (between $\psi^\dagger(t',\vec{x}')$ and $\psi(t',\vec{x}')$). Consequently, the equation for $u_k(t,\vec{x})$ becomes

$$(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M})u_k(t,\vec{x}) = \int d^3q d^3x' dt' V(t-t';\vec{x}-\vec{x}')\, u_q(t,\vec{x}) \times$$

$$\times\, u_q^*(t',\vec{x}')u_k(t',\vec{x}') \qquad \text{(III.5)}$$

It is obvious from (III.5) that although the original field equation (III.1) was non-linear, because the field

is either only creation or only annihilation, the one-particle wavefunction does not couple to any other wavefunction.

Upon making the Ansatz

$$u_k(t,\vec{x}) = Ae^{-i(E_k t - \vec{k}\cdot\vec{x})}, \qquad \text{(III.6)}$$

eqn. (III.5) is converted into an integral equation for E_k, which can be written symbolically as

$$E_k = \frac{k^2}{2M} + (2\pi)V_k \qquad \text{(III.7)}$$

It is, of course, not at all obvious from (III.7) that this is an integral equation; in fact the energy just seems to have been shifted by an amount proportional to V_k. However, V_k is actually given by

$$V_k = \int d^3q \, \tilde{\tilde{V}}[E_k - E_q; \vec{k}-\vec{q}] \qquad \text{(III.8)}$$

with

$$\tilde{\tilde{V}}(\omega,\vec{k}) = \frac{1}{(2\pi)^4}\int dt \, d^3x \, V(t,\vec{x}) e^{-i(\omega t - \vec{k}\cdot\vec{x})} \qquad \text{(III.9)}$$

Therefore, as V_k itself is an integral involving E_k, eqn. (III.7) is obviously a non-linear integral equation for E_k.

(Note also that the factor 2π in (III.7) arises from the use of the ETCR (or ETACR), which gives $A^2 = (2\pi)^{-3}$.)

Instead of actually solving for E_k from (III.7), which is a formidable task, what one does is to insert the "free" value of the energy i.e.

$$E_k = \frac{k^2}{2M}$$

into the expression for V_k. Then V_k is reduced to purely an integral term; as it is the difference between the "total" and "free" energies for a one-particle system, it is identified as the self-energy. It should be mentioned, at this juncture, that this definition of self-energy is consistent with the corresponding situation for the static case, where $V(t-t';\vec{x}-\vec{x}') = \delta(t-t')V_o(\vec{x}-\vec{x}')$, which has also been discussed by Capri[22].

To conclude, the one-particle wavefunctions are, in fact plane wave solutions, with the energy being defined by eqn. (III.7). Since the $u_k(t,\vec{x})$ form a complete set, one need not look for other solutions to the one-particle equation.

III.2 Two Particle States

Because of the two possible types of states that could occur in this case, viz. scattering states $|\vec{p}_1\ \vec{p}_2>$ and bound states $|E_n\ \vec{p}>$, one naturally has two types of wavefunctions, and consequently two equations governing their behaviour. Scattering and bound state wavefunctions are defined as follows:

$$F_{p_1p_2}(t,\vec{x},\vec{z}) = <0|\psi(t,\vec{z})\psi(t,\vec{x})|\vec{p}_1\vec{p}_2> \qquad \text{(III.10)}$$

$$F_{pn}(t,\vec{x},\vec{z}) = <0|\psi(t,\vec{z})\psi(t,\vec{x})|E_n\vec{p}> \qquad \text{(III.11)}$$

One can derive, as in ref. 22, equations for the objects $F_{p_1p_2}(t,\vec{x},\vec{z})$ and $F_{pn}(t,\vec{x},\vec{z})$. However, the equations one eventually wants are for functions of simpler structure. These functions are obtained as follows: as one is interested in stationary states, one writes

$$F_{p_1p_2}(t,\vec{x},\vec{z}) = e^{-iE(\vec{p}_1,\vec{p}_2)t} f_{p_1p_2}(\vec{x},\vec{z}) \qquad \text{(III.12)}$$

$$F_{pn}(t,\vec{x},\vec{z}) = e^{-iE_n(\vec{p})t} f_{pn}(\vec{x},\vec{z}) \qquad \text{(III.13)}$$

From this point on, the one-particle energy E_k is assumed to be just the free part, i.e. self-energy is neglected. Then, the energies $E(\vec{p}_1,\vec{p}_2)$ and $E_n(\vec{p})$ are written as

$$E(\vec{p}_1,\vec{p}_2) = E_{p_1} + E_{p_2} = \frac{p_1^2+p_2^2}{2M} \qquad \text{(III.14)}$$

$$E_n(p) = \frac{p^2}{4M} - E_n \ , \qquad \text{(III.15)}$$

E_n being the binding energy of the nth bound state. Further, the centre of mass motion may be separated out as follows

$$f_{p_1p_2}(\vec{x},\vec{z}) = e^{i(\vec{p}_1+\vec{p}_2)\cdot(\frac{\vec{x}+\vec{z}}{2})} f_{\vec{p}_1-\vec{p}_2}(\vec{x}-\vec{z}) \qquad \text{(III.16)}$$

with a similar relation for the bound state. The equations one eventually obtains are for the functions $f_q(\vec{r})$ where $\vec{q}$ and $\vec{r}$ are related to the relative momentum and relative co-ordinate variables respectively. Substituting eqns. (III.12) to (III.16) into the equations for $F_{p_1p_2}$ and F_{pn}, one is led, after fairly simple but tedious algebra, to the following pair of Schrödinger equations, in the centre of momentum frame:

$$\begin{aligned}(\nabla_r^2 + p^2) f_p(\vec{r}) &= 2\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}') f_p(\vec{r}') \\ &+ M \int d^3r' \left[\int d^3q\ V_{pq}(\vec{r}-\vec{r}') f_q(\vec{r}) f_q^*(\vec{r}')\right] f_p(\vec{r}') \\ &+ M \int d^3r' \left[\sum_n V_{pn}(\vec{r}-\vec{r}') f_n(\vec{r}) f_n^*(\vec{r}')\right] f_p(\vec{r}')\end{aligned} \qquad \text{(III.17)}$$

$$\begin{aligned}(\nabla_r^2 - ME_\ell) f_\ell(\vec{r}) &= 2\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}') f_\ell(\vec{r}') \\ &+ M \int d^3r' \left[\int d^3q\ V_{\ell q}(\vec{r}-\vec{r}') f_q(\vec{r}) f_q^*(\vec{r}')\right] f_\ell(\vec{r}') \\ &+ M \int d^3r' \left[\sum_n V_{\ell n}(\vec{r}-\vec{r}') f_n(\vec{r}) f_n^*(\vec{r}')\right] f_\ell(\vec{r}')\end{aligned} \qquad \text{(III.18)}$$

The V's are given by:

$$V_o(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3k\, d^3q\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\tilde{V}}[\tfrac{q^2-k^2}{2M}\,;\, \vec{q}-\vec{k}] \qquad \text{(III.19)}$$

$$V_{pq}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\, \tilde{\tilde{V}}[\tfrac{q^2-p^2}{M} + \tfrac{k^2}{4M}\,;\vec{k}] \qquad \text{(III.20)}$$

$$V_{pn}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\, \tilde{\tilde{V}}[\tfrac{k^2-4p^2}{4M} - E_n;\vec{k}] \qquad \text{(III.21)}$$

$$V_{\ell q}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\, \tilde{\tilde{V}}[\tfrac{q^2}{M} + \tfrac{k^2}{4M} + E_\ell;\vec{k}] \qquad \text{(III.22)}$$

$$V_{\ell n}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\, \tilde{\tilde{V}}[\tfrac{k^2}{4M} + E_\ell - E_n;\vec{k}] \qquad \text{(III.23)}$$

(The above equations differ from the corresponding ones in ref. (22), because the latter contains errors.) In addition, if one works out eqn. (III.3), it can be seen that the ETCR (or ETACR) reduces to the following condition:

$$\int d^3q\, f_q(\vec{r}) f_q^*(\vec{r}') + \sum_n f_n(\vec{r}) f_n^*(\vec{r}') = \frac{1}{(2\pi)^3}[\delta(\vec{r}-\vec{r}') \pm \delta(\vec{r}+\vec{r}')] \qquad \text{(III.24)}$$

What has been done here is the following: starting from the original equation for the field ψ, equations have been derived for the possible two-particle states. Equations (III.17) and (III.18) form a pair of coupled, non-linear Schrödinger equations which describes the wavefunctions for the stationary states of two particles interacting via $V(t-t';\vec{x}-\vec{x}')$. However, the coupling is only between the

two-particle states themselves, and between the two-particle and one-particle states through the term containing $V_o(\vec{r}-\vec{r}')$, i.e. there is no coupling to higher-particle wavefunctions. A certain amount of decoupling has therefore taken place. In addition, the terms in the square brackets in eqns. (III.17) and (III.18) are interpreted as "potentials" within the Schrödinger equation, and are, in general, explicitly non-local and energy-dependent. This is purely a consequence of the original equation of motion, and if one has a model for the N-N interaction governed by such an equation, one will be led quite naturally to non-local, energy-dependent potentials.

The solutions $f_p(\vec{r})$ and $f_n(\vec{r})$ are constrained further by requirement (III.24); it is a completeness condition on the two-particle solutions, which is, in this form, a normalisation condition.

One point that has to be mentioned at this stage has to do with the self-energy term. One has identified the term

$$2\pi M \int d^3r' V_o(\vec{r}-\vec{r}') f(\vec{r}')$$

in eqns. (III.17) and (III.18) with self-energy. This identification becomes obvious when one sees that $V_o(\vec{x})$ and V_k are related by

$$V_o(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3k \, V_k \, e^{i\vec{k}\cdot\vec{x}}$$

Before calculations can be done, the self-energy has to be explicitly subtracted out. This will be discussed further in the next section.

What this technique does, therefore, is the following: starting from a non-relativistic field equation of the type (III.1), one derives a set of coupled, non-linear n-particle Schrödinger equations. Because of the non-relativistic nature of the model, these equations can be partially decoupled, though they remain non-linear. When, therefore, one approaches the N-N problem, one will be led to equations describing the wavefunctions of two-nucleon states, in both scattering and bound state channels. From these equations, physical properties of the N-N system, such as phase shifts, binding energies, effective ranges, etc. can, in principle, be extracted. As they stand, however, (III.17) and (III.18) are obviously intractable, and they will first have to be simplified.

The two models to be considered are the scalar and the pseudoscalar models. In both these models, the equation for the nucleon field can be reduced to the form as given by (III.1), and so the technique described above goes through in a straightforward manner.

IV THE SCALAR MODEL

Field theory describes the interaction between nucleons as being transmitted via particle exchange. Symmetry and invariance requirements dictate the particle exchanges possible. In the two models to be considered, it is assumed that meson fields act as the intermediary between the nucleons. The difference between the two models is in the type of meson exchanged. In the scalar model[23], there is no spin or charge; obviously, such a model does not give a true description of the physics. In fact, "truly" scalar mesons, which are described by equations derivable from a Lagrangian, give a repulsive potential. Therefore, in order to obtain an attractive potential, and hence simulate pseudoscalar mesons, one uses, in fact, field equations not derivable from a Lagrangian. The pseudoscalar model, on the other hand, is more "physical", and does include spin and isospin. However, because of the far more complicated structure of the pseudoscalar theory, the scalar model is considered first. In this way, one is able to develop more easily, the techniques required, and also to gain insight into the computational problems that exist, and the complications that may arise.

The equations for the coupled nucleon and meson fields in the scalar theory are

$$(i\frac{\partial}{\partial t} + \frac{\nabla}{2M})\psi(t,\vec{x}) = g\psi(t,\vec{x})\phi(t,\vec{x}) \qquad \text{(IV.1)}$$

$$(\Box + m^2)\phi(t,\vec{x}) = - g\psi^{\dagger}(t,\vec{x})\psi(t,\vec{x}). \qquad \text{(IV.2)}$$

Here, ψ, ϕ are the nucleon and meson fields respectively, and g is the coupling constant. Eqn. (IV.2) can be solved, at least formally, and the solution written in integral form as follows:

$$\phi(t,\vec{x}) = - g \int \bar{\Delta}(t-t';\vec{x}-\vec{x}')\psi^{\dagger}(t',\vec{x}')\psi(t',\vec{x}')dt'd^3x'. \qquad \text{(IV.3)}$$

There is no inhomogeneous term in (IV.3). One reason for this is that one assumes that there are no free pions - all the pions created are absorbed. Furthermore, our interest is restricted mainly to elastic N-N interactions. In fact, this is also the reason for the choice of $\bar{\Delta}$ as the appropriate Green's function. Eqn. (IV.3), when substituted into (IV.1) gives an equation in ψ alone; viz.

$$(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2M})\psi(t,\vec{x}) = - g^2 \int \bar{\Delta}(t-t';\vec{x}-\vec{x}')\psi(t,\vec{x})\psi^{\dagger}(t',\vec{x}') \times$$

$$\times \psi(t',x')\, dt'\, d^3x' . \qquad \text{(IV.4)}$$

This has been achieved because the "source" term for the pion field is dependent only upon the nucleon field. Although (IV.4) is non-linear, it is independent (at least explicitly) of the meson field, and is of the type given by eqn. (III.1). The potential, in this case, is [24]

$$V(t,\vec{x}) = -g^2\bar{\Delta}(t,\vec{x}) = \frac{g^2}{(2\pi)^4}\int d\omega\, d^3k\, \frac{e^{i(\omega t-\vec{k}\cdot\vec{x})}}{\omega^2-k^2-m^2}$$

Therefore,

$$\tilde{\tilde{V}}(\omega,\vec{k}) = \frac{g^2}{(2\pi)^4}\,\frac{1}{\omega^2-k^2-m^2}\,. \qquad \text{(IV.6)}$$

In this model, physical states are defined, as outlined in Section III. Therefore, one-particle states are defined as

$$u_k(t,\vec{x}) = Ae^{i(E_k t-\vec{k}\cdot\vec{x})} \qquad \text{(IV.7)}$$

and this leads to the integral equation for E_k:

$$E_k = \frac{k^2}{2M} -$$

$$- g^2|A|^2\int d^3x'dt'd^3q\bar{\Delta}(t-t';\vec{x}-\vec{x}')e^{-i(E_q-E_k)(t-t')}\,e^{i(\vec{q}-\vec{k})\cdot(\vec{x}-\vec{x}')}$$

$$= \frac{k^2}{2M} + \frac{g^2}{(2\pi)^3}\int \frac{d^3q}{(E_k-E_q)^2-(\vec{k}-\vec{q})^2-m^2}\,. \qquad \text{(IV.8)}$$

The ETCR have been used to obtain $|A|^2 = (2\pi)^{-3}$. The second term in (IV.8) is what one refers to as the self-energy.

IV.1 The N-N Equations

In the 2-nucleon system, there is only one bound state; namely, the deuteron. So, the label n used in Section III (for the number of bound states) will be dropped, as n = 1. The scattering states, therefore, are of the type $|\vec{k}_1\ \vec{k}_2>$, and the bound state is $|E\ \vec{k}>$. As in Section III, one can obtain equations for the corresponding wavefunctions. They will be identically the same as eqns. (III.17) and (III.18), with the structure of the $\tilde{\tilde{V}}$'s given by eqn. (IV.6).

At this point, one makes an additional assumption - one assumes that the contribution to the non-local potentials from the intermediate scattering states is dominant, and that the bound state contributions may be neglected. One reason for this is the following: in the pseudoscalar model, it will be seen that in the equation for the deuteron wavefunction, the only intermediate states that appear are scattering states, i.e. the bound state does not couple to itself. Therefore, as one is interested in the scalar model mainly as a guide to handling the more physical, pseudoscalar model, one is quite justified in dropping the bound

state contribution in the former. There is also another reason: if one solves the Schrödinger equation with a δ-function (or zero-width) potential, for both scattering states and bound states, one can then evaluate integrals of the type

$$\int d^3q\ V_{\alpha\beta}(\vec{r}-\vec{r}')f_q(\vec{r})f_q^*(\vec{r}')$$

where the f_q's are these solutions. ($V_{\alpha\beta}$ stands for the possible V's as tabulated in (III.19) to (III.23).) It can then be seen that contributions from intermediate bound states are small compared to those from the scattering states.

Upon dropping these terms, one gets the following:

$$(\nabla_r^2+p^2)f_p(\vec{r}) = 2\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}')f_p(\vec{r}')$$

$$+ M \int d^3q\ d^3r'\ V_{pq}(\vec{r}-\vec{r}')f_q(\vec{r})f_q^*(\vec{r}')f_p(\vec{r}') \qquad \text{(IV.10)}$$

$$(\nabla_r^2-ME)f_b(\vec{r}) = 2\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}')f_b(\vec{r}')$$

$$+ M \int d^3q\ d^3r'\ V_{bq}(\vec{r}-\vec{r}')f_q(\vec{r})f_q^*(\vec{r}')f_b(\vec{r}') \qquad \text{(IV.11)}$$

with

$$V_{pq}(\vec{x}) = (2\pi)^4 \int d^3k \; e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \; \tilde{\tilde{V}}\left[\frac{4q^2-4p^2+k^2}{4M} \; ; \vec{k}\right] \qquad \text{(IV.12)}$$

$$V_{bq}(\vec{x}) = (2\pi)^4 \int d^3k \; e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \; \tilde{\tilde{V}}\left[\frac{4q^2+4ME+k^2}{4M} \; ; \vec{k}\right] . \qquad \text{(IV.13)}$$

For the energy range one is interested in, where p is small, these potentials are almost identical i.e. for low energies, both types of states see the same potential. In addition, (IV.10) and (IV.11) are related through the substitution

$$p^2 \quad \leftrightarrow \quad -ME$$

There are two points to be taken into consideration here - firstly, eqns. (IV.10) and (IV.11) are non-linear in the wavefunction, and therefore difficult to solve. The way this is achieved here is, in a sense, iterative; we only carry out this iteration to first order, and so the intermediate states are assumed to be free states. Therefore, properly normalised and symmetrised plane wave solutions are inserted into the integrals in (IV.10) and (IV.11). Hence, the equations have been linearised. The second point has to do with the self-energy term, which is still present in the equations. Once again, one uses the static model,

which has been solved by Capri, as the guideline. Twice the one-particle self energy term, i.e. a term $4\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}')f(\vec{r}')$ is subtracted from the right hand side of the equations. It will become apparent later, that such a subtraction is necessary for the equations to be consistent.

When these points are taken into consideration, the equations become

$$(\nabla_r^2+p^2) f_p(\vec{r}) = - 2\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}') f_p(\vec{r}')$$

$$+ M \int d^3r'\ d^3q\ V_{pq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') f_p(\vec{r}') \qquad \text{(IV.14)}$$

$$(\nabla_r^2-ME) f_b(\vec{r}) = - 2\pi M \int d^3r'\ V_o(\vec{r}-\vec{r}') f_b(\vec{r}')$$

$$+ M \int d^3r'\ d^3q\ V_{bq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') f_b(\vec{r}') \qquad \text{(IV.15)}$$

where the superscript $^{(o)}$ stands for free solutions. These equations will be examined in both configuration-space and momentum-space.

IV.2 Configuration Space Potentials

The ground state of the deuteron is a spin triplet, isospin singlet state. Therefore, for the wavefunction to stay odd under particle interchange, the spatial part has to be even under parity. Therefore the intermediate wavefunctions have to be symmetric in the spatial coordinate. So we write

$$f_p^{(o)}(\vec{r}) = C[e^{i\vec{p}\cdot\vec{r}} + e^{-i\vec{p}\cdot\vec{r}}] \qquad \text{(IV.16)}$$

where C is determined from the ETCR (eqn. III.24), and turns out to be

$$C = \frac{1}{\sqrt{2}(2\pi)^3} \qquad \text{(IV.17)}$$

Non-local potentials $V_p^{NL}(\vec{r},\vec{r}')$ and $V_b^{NL}(\vec{r}-\vec{r}')$ are defined as follows:

$$V_p^{NL}(\vec{r},\vec{r}') = -2\pi M\, V_o(\vec{r}-\vec{r}') + M \int d^3q\, V_{pq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') \qquad \text{(IV.18)}$$

$$V_b^{NL}(\vec{r},\vec{r}') = -2\pi M\, V_o(\vec{r}-\vec{r}') + M \int d^3q\, V_{bq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') \qquad \text{(IV.19)}$$

The superscript NL stands for "non-local"; with these definitions, (IV.14) and (IV.15) become

$$(\nabla_r^2+p^2) f_p(\vec{r}) = \int V_p^{NL}(\vec{r},\vec{r}') f_p(\vec{r}') d^3r' \qquad \text{(IV.20)}$$

$$(\nabla_r^2-ME) f_b(\vec{r}) = \int V_b^{NL}(\vec{r},\vec{r}') f_b(\vec{r}') d^3r' \qquad \text{(IV.21)}$$

Eqns. (IV.20) and (IV.21) are just Schrodinger equations with non-local, energy-dependent potentials which describe the scattering and bound states of the 2-nucleon system. It would, of course, be very desirable to have the V^{NL}'s explicitly as functions of $\vec{r}$ and $\vec{r}'$; however, the actual task is formidable. Consider first, the object $Q_p(\vec{r},\vec{r}')$ which is

$$Q_p(\vec{r},\vec{r}') = \int d^3q\, V_{pq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') \qquad \text{(IV.22)}$$

$$= C^2 \int d^3q\, V_{pq}(\vec{r}-\vec{r}') [e^{i\vec{q}\cdot\vec{r}}+e^{-i\vec{q}\cdot\vec{r}}][e^{-i\vec{q}\cdot\vec{r}'}+e^{i\vec{q}\cdot\vec{r}'}]$$

$$= 2C^2 \int d^3q\, V_{pq}(\vec{r}-\vec{r}') [e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}+e^{i\vec{q}\cdot(\vec{r}+\vec{r}')}].$$

Then, one can write

$$V_p^{NL}(\vec{r},\vec{r}') = MQ_p(\vec{r},\vec{r}') - 2\pi M\, V_o(\vec{r}-\vec{r}')$$

and

$$V_b^{NL}(\vec{r},\vec{r}') = MQ_b(\vec{r},\vec{r}') - 2\pi M\, V_o(\vec{r},\vec{r}'),$$

$Q_b(\vec{r},\vec{r}')$ being analogous to $Q_p(\vec{r},\vec{r}')$.

Using the results of Appendix A, where these integrals are evaluated, one gets that

$$Q_p(\vec{r},\vec{r}') = \frac{16C^2(2\pi)^3Mg^2}{|\vec{r}-\vec{r}'|}\,\mathrm{Re}\int_0^\infty dk\,\frac{k\,\sin(\frac{k}{2}|\vec{r}-\vec{r}'|)}{\sqrt{k^2+m^2}}\times$$

$$\times\left(\frac{e^{-[\frac{k^2}{4}-p^2-M\sqrt{k^2+m^2}]^{1/2}|\vec{r}-\vec{r}'|}-e^{-[\frac{k^2}{4}-p^2+M\sqrt{k^2+m^2}]^{1/2}|\vec{r}-\vec{r}'|}}{|\vec{r}+\vec{r}'|}+\right.$$

$$\left.\frac{e^{-[\frac{k^2}{4}-p^2-M\sqrt{k^2+m^2}]^{1/2}|\vec{r}+\vec{r}'|}-e^{-[\frac{k^2}{4}-p^2+M\sqrt{k^2+m^2}]^{1/2}|\vec{r}+\vec{r}'|}}{|\vec{r}+\vec{r}'|}\right) \qquad \text{(IV.23)}$$

Because of the highly complicated structure of the arguments within the exponentials in eqn. (IV.23), the k-integral cannot be evaluated in closed form analytically - one can look at the problem equally well in momentum space, where equations for the T-matrix may be derived. As physical quantities are easily obtained from the T-matrix, the configuration space potentials will be left as in eqn. (IV.23).

IV.3 Momentum Space Considerations

Eqn. (IV.14), the equation for the scattering wavefunction $f_p(\vec{r})$ reads:

$$(\nabla_r^2+p^2) f_p(\vec{r}) = -2\pi M \int d^3r'\, V_o(\vec{r}-\vec{r}') f_p(\vec{r}')$$

$$+ M \int d^3r'\, d^3q\, V_{pq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') f_p(\vec{r}').$$

In momentum space, the corresponding wavefunctions are defined by:

$$f_p(\vec{r}) = \int d^3k\, e^{-i\vec{k}\cdot\vec{r}} \psi_p(\vec{k}) \qquad \text{(IV.24)}$$

$$f_b(\vec{r}) = \int d^3k\, e^{-i\vec{k}\cdot\vec{r}} \psi_b(\vec{k}). \qquad \text{(IV.25)}$$

The equations for $\psi_p(\vec{k})$ and $\psi_b(\vec{k})$ are obtained in a straightforward manner from the configuration space equations (Appendix B). For $\psi_p(\vec{k})$, in fact, the equation is

$$(p^2-k^2)\psi_p(\vec{k}) = -2\pi M \int d^3q\, \tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M};\ \vec{q}+\vec{k}\right] \psi_p(\vec{k})$$

$$+ 8M(2\pi)^7 \int d^3k_1\, d^3k_2\, d^3k_3\, \tilde{\tilde{V}}\left[\frac{(\vec{k}_2-\vec{k}_3)^2+k_1^2-p^2}{4M};\ 2(\vec{k}_3-\vec{k}_2)\right] \times$$

$$\times \psi_{k_1}^{(o)}(\vec{k}+\vec{k}_2-\vec{k}_3) \psi_{k_1}^{(o)*}(\vec{k}_2) \psi_p(\vec{k}_3).$$

(IV.26)

To obtain the free intermediate wavefunctions $\psi_p^{(o)}(\vec{k})$, one uses eqns. (IV.24), (IV.16) and (IV.17); these give

$$\psi_p^{(o)}(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[\delta(\vec{p}+\vec{k}) + \delta(\vec{p}-\vec{k})\right] \qquad \text{(IV.27)}$$

Substituting (IV.27) into (IV.26) yields

$$(p^2-k^2)\psi_p(\vec{k}) = -2\pi M \int d^3q\, \tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M}\,;\,\vec{q}+\vec{k}\right]\psi_p(\vec{k})$$

$$+ 2\pi M \int d^3q\, \tilde{\tilde{V}}\left[\frac{q^2+k^2-2p^2}{2M}\,;\,\vec{q}+\vec{k}\right](\psi_p(\vec{k})+\psi_p(\vec{q})) \qquad \text{(IV.28)}$$

the derivation of which is in App. B. One can now combine the coefficients of $\psi_p(\vec{k})$ accordingly, and with a function F(p,k) defined as

$$F(p,k) = 2\pi M \int d^3q \left(\tilde{\tilde{V}}\left[\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right]-\tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M};\vec{q}+\vec{k}\right]\right) \qquad \text{(IV.29)}$$

eqn. (IV.28) becomes:

$$[p^2-k^2-F(p,k)]\psi_p(\vec{k}) = 2\pi M \int d^3q\, \tilde{\tilde{V}}\left[\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right]\psi_p(\vec{q}). \qquad \text{(IV.30)}$$

This is, therefore the integral equation for the wavefunction $\psi_p(\vec{k})$ in momentum space (analogous to the integrodifferential eqn. (IV.20) in configuration space). F(p,k) contains the self-energy, and one sees that when k=p, from eqn. (IV.29),

F(p,k) vanishes. The relevance of this becomes obvious in Section IV.5, where the self-energy subtraction is examined in closer detail.

One object of transforming to momentum space is to enable one to obtain an expression for the T-matrix. In terms of the wavefunction $\psi_p(\vec{k})$, the full T-matrix is defined by[25]

$$\psi_p(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}[\delta(\vec{p}+\vec{k})+\delta(\vec{p}-\vec{k})] + \frac{M}{\sqrt{2}(2\pi)^3}\frac{T(\vec{k},\vec{p},p^2)+T(-\vec{k},\vec{p},p^2)}{p^2-k^2+i\eta} \tag{IV.31}$$

The T's in eqn. (IV.31) are <u>half-off shell</u> quantities, the label p^2 indicating that $\vec{p}$ is on-shell. From now on, the p^2 will be suppressed, and we shall write the T-matrix as $T(\vec{k},\vec{p})$.

Experimental quantities, like phase shifts, etc., are related to the <u>on-shell</u> part of the T-matrix, viz. $T(\vec{p},\vec{p})$[26]. Substituting (IV.31) into (IV.30) gives the equation for the T-matrix:

$$T(\vec{k},\vec{p}) = \pi\frac{p^2-k^2}{p^2-k^2-F(\vec{p},\vec{k})}\left[\tilde{\tilde{v}}\left(\frac{k^2-p^2}{2M};\vec{k}-\vec{p}\right) + \tilde{\tilde{v}}\left(\frac{k^2-p^2}{2M};\vec{k}+\vec{p}\right)\right]$$

$$+ M\frac{p^2-k^2}{p^2-k^2-F(\vec{p},\vec{k})}\int\frac{d^3q}{p^2-q^2+i\eta}\tilde{\tilde{v}}\left[\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right]T(\vec{q},\vec{p}) \tag{IV.32}$$

with

$$T(\vec{k},\vec{p}) = T(-\vec{k},\vec{p}) \ .$$

The equation for any given partial wave may be obtained from eqn. (IV.32); however, it is easier, in fact, to do this using the equation for $\psi_p(\vec{k})$ (eqn. (IV.30)) (see Appendix C). Eqn. (C16) for the s-wave T-matrix, T_o is:

$$T_o(k,p) = \frac{g^2}{2(2\pi)^2} \frac{M(p,k)}{pk} \ln\left|\frac{(k^2-p^2)^2-4M^2(k-p)^2-4m^2M^2}{(k^2-p^2)^2-4M^2(k+p)^2-4m^2M^2}\right|$$

$$+ \frac{Mg^2}{2(2\pi)^2} \frac{M(p,k)}{k} \int_0^\infty \frac{qdq}{p^2-q^2+i\eta} \ln\left|\frac{(q^2+k^2-2p^2)^2-4M^2(q-k)^2-4m^2M^2}{(q^2+k^2-2p^2)^2-4M^2(q+k)^2-4m^2M^2}\right| T_o(q,p)$$

(IV.33)

This is, of course, an integral equation for T_o; although one can obtain a great deal of information from the <u>on-shell</u> part of T_o, it is obvious from (IV.33) that one will never have the on-shell part until one has solved (IV.33) i.e. one needs the <u>full</u> T_o before physical quantities can be extracted. However, by going into momentum space, one has reduced the integrodifferential eqn. (IV.10) into a purely integral one, viz. (IV.33).

IV.4 Born Approximation

The most frequently encountered approximation for solving the Schrodinger equation is that of Born[27]. It amounts to a series expansion in "powers" of the interaction potential. The first term in such an expansion is referred to as the Born approximation. If one considers a Yukawa potential of the type

$$V(r) = \frac{-V_o}{m} \frac{e^{-mr}}{r} ,$$

it can be shown that the first Born approximation gives

$$T^{(1)} = - \frac{V_o}{(2\pi)^2 m} \left[\frac{1}{m^2+(\Delta\vec{k})^2} + \frac{1}{m^2+(\Sigma\vec{k})^2} \right] . \qquad \text{(IV.34)}$$

In (IV.34), $\Sigma\vec{k}$ and $\Delta\vec{k}$ are the total momentum and momentum transfer respectively, and the numerical factor comes from normalisation.

In our case, the Born term can be derived in analogous fashion (i.e. from the inhomogeneous term in the equation for the full (non-partial wave decomposed) T-matrix (eqn. IV.32)). Then, one goes on to the mass shell by making $|\vec{k}| = |\vec{p}|$. This gives:

$$T^{(1)}(p) = \pi \lim_{k \to p} \frac{p^2-k^2}{p^2-k^2-F(\vec{p},\vec{k})} [\tilde{\tilde{V}}(o,\vec{k}-\vec{p}) + \tilde{\tilde{V}}(o,\vec{k}+\vec{p})] . \qquad \text{(IV.35)}$$

Dropping the self energy, this becomes

$$T^{(1)}(p) = -\frac{\pi g^2}{(2\pi)^4}\left[\frac{1}{m^2+|\vec{k}-\vec{p}|^2} + \frac{1}{m^2+|\vec{k}+\vec{p}|^2}\right]. \quad \text{(IV.36)}$$

Comparing (IV.34) and (IV.36), we see that the Born approximation for the Yukawa potential is reproduced with the coupling constant given by the relation

$$\frac{V_o}{m}\frac{1}{(2\pi)^2} = \frac{\pi g^2}{(2\pi)^4}$$

$$\therefore \quad \frac{g^2}{(2\pi)^3} = \frac{V_o}{m}\frac{1}{2\pi^2} \; . \quad \text{(IV.37)}$$

For the Yukawa potential of the type we have considered, it is possible to obtain the deuteron properties (e.g. binding energy)[28] if one makes

$$V_o = 41.5 \text{ MeV}$$

$$m = 1.413 \text{ f}^{-1} \; .$$

Substituting these values into (IV.37) gives

$$\frac{g^2}{(2\pi)^3} = 0.0164 \quad \text{(IV.38)}$$

In our bound state calculation (to be discussed in Section IV.6), a variational approach was used to obtain a maximum value for the binding energy E_b. Therefore the corresponding coupling constant obtained will in fact be an upper limit to its true value, and hence corresponds to stronger binding. The value we obtained was $g^2 = 4.6$, or

$$\frac{g^2}{(2\pi)^3} = 0.0185$$

This value is in fact larger than that obtained from the Yukawa potential, viz. 0.0164, but at the same time is close enough to it to indicate good agreement between this model and the Yukawa model, at least within the Born approximation. At the same time, these two values seem to indicate internal consistency of the model.

IV.5 Self-Energy Subtraction

In Section IV.1, the self-energy term was taken care of in a very particular way. In fact, exactly twice the one-particle self energy term that appeared in the original equation, viz. (IV.10), was subtracted out. In this Section, it is shown that this is exactly what has to be done in fact, for the equation for the T-matrix to be free of inconsistencies. In order to show this, one considers eqn. (C16) for the s-wave part of T, which is

$$[p^2-k^2-F(p,k)]T_o(k,p) =$$

$$\frac{g^2}{2(2\pi)^2}\,\frac{p^2-k^2}{kp}\,\ell n\left|\frac{(k^2-p^2)^2-4M^2(k-p)^2-4M^2m^2}{(k^2-p^2)^2-4M^2(k+p)^2-4M^2m^2}\right|$$

$$+\frac{Mg^2}{2(2\pi)^2}\,\frac{p^2-k^2}{k}\int_0^\infty \frac{qdq}{p^2-q^2+i\eta}\times$$

$$\times\,\ell n\left|\frac{(q^2+k^2-2p^2)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2-2p^2)^2-4M^2(q+k)^2-4M^2m^2}\right| T_o(q,p).$$

(IV.39)

If one assumes that a quantity which is α x the self-energy is to be subtracted i.e. if, after subtraction, eqn. (IV.10) becomes

$$(\nabla_r^2+p^2)f_p(\vec{r}) = (2\pi M-\alpha)\int d^3r'\; V_o(\vec{r}-\vec{r}')f_p(\vec{r}') +$$

$$+ M\int d^3q\, d^3r'\; V_{pq}(\vec{r}-\vec{r}')f_q(\vec{r})f_q^*(\vec{r}')f_p(\vec{r}')$$

one is led to the following equation in momentum space

$$(p^2-k^2)\psi_p(\vec{k}) = (2\pi M-\alpha)\int d^3q\;\tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M};\ \vec{q}+\vec{k}\right]\psi_p(\vec{k})$$

$$+ 2\pi M\int d^3q\;\tilde{\tilde{V}}\left[\frac{q^2+k^2-2p^2}{2M};\ \vec{q}+\vec{k}\right](\psi_p(\vec{k}) + \psi_p(\vec{q})).$$

(IV.40)

This becomes obvious when one looks at (IV.28). So the quantity F(p,k) as defined by (IV.29) is modified to read

$$F(p,k) = 2\pi M \int d^3q \left(\tilde{\tilde{V}}\left[\frac{q^2+k^2-2p^2}{2M}; \vec{q}+\vec{k}\right] + \tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M}; \vec{q}+\vec{k}\right] \right)$$

$$- \alpha \int d^3q \, \tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M}; \vec{q}+\vec{k}\right] . \qquad \text{(IV.41)}$$

If now, one considers the on-shell limit of eqn. (IV.39), i.e. letting $p \to k$, one gets

$$F(k,k)T_o(k,k) = 0$$

which implies that, since the on-shell T-matrix can never be zero for all momenta,

$$F(k,k) = 0 \qquad \text{(IV.42)}$$

Eqns. (IV.41) and (IV.42) imply that

$$(4\pi M-\alpha) \int d^3q \, \tilde{\tilde{V}}\left[\frac{q^2-k^2}{2M}; \vec{q}+\vec{k}\right] = 0$$

So one is led to the conclusion that if $V_o(\vec{r}-\vec{r}')$ is to be interpreted as self-energy, then $\alpha = 4\pi M$, and in fact twice the one-particle self energy has to be subtracted. As mentioned previously, this is consistent with the subtraction in the static case, where it is obvious. It amounts to

subtracting twice the single-particle self-energy, as it is a two-particle system.

IV.6 The Bound State

In Section IV.1, it was pointed out that the bound state potential is obtained from the scattering state potential by the replacement

$$p^2 \to -ME$$

The s-wave bound state equation can therefore be written down quite easily, by using eqn. (C9). One first defines

$$V_b(\vec{q},\vec{k}) = \tilde{\tilde{V}}\left[\frac{q^2+k^2+2ME}{2M};\ q+k\right] \qquad \text{(IV.43)}$$

and

$$F_b(\vec{k}) = 2\pi M \int d^3q\ [V_b(\vec{q},\vec{k}) - V_k(\vec{q},\vec{k})] \qquad \text{(IV.44)}$$

in analogy with (C1) and (C6) respectively. Then the required equation becomes

$$[k^2+ME+F_b(k)]\psi_b^{(o)}(k) = \qquad \text{(IV.45)}$$

$$-\frac{Mg^2}{2(2\pi)^2 k}\int_0^\infty qdq\ \ell n\left|\frac{(q^2+k^2+2ME)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2+2ME)^2-4M^2(q+k)^2-4M^2m^2}\right|\psi_b^{(o)}(q)$$

or,

$$ME\psi_b^{(o)}(k) = -k^2\psi_b^{(o)}(k) - F_b(k)\psi_b^{(o)}(k) -$$

$$- \frac{Mg^2}{2(2\pi)^2 k}\int_0^\infty qdq\ \ell n\left|\frac{(q^2+k^2+2ME)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2+2ME)^2-4M^2(q+k)^2-4M^2m^2}\right|\psi_b^{(o)}(q) \qquad (IV.46)$$

In principle, one would use a variational technique to determine the binding energy E, given the coupling constant g^2. However, what is done here is to determine the value of g^2 for which E is <u>both</u> a maximum, and also close to the physical binding energy. So one needs to maximise

$$ME = -\left(\int_0^\infty k^4|\psi_b^{(o)}(k)|^2dk + \int^\infty k^2F_b(k)|\psi_b^{(o)}(k)|^2dk +\right.$$

$$+ \frac{Mg^2}{2(2\pi)^2}\int_0^\infty kdk\int_0^\infty qdq\ \ell n\left|\frac{(q^2+k^2+2ME)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2+2ME)^2-4M^2(q+k)^2-4M^2m^2}\right| \times$$

$$\left. \times\ \psi_b^{(o)}(k)\psi_b^{(o)}(q)\right) \div$$

$$\div \int_0^\infty k^2|\psi_b^{(o)}(k)|^2dk\ . \qquad (IV.47)$$

One form of the deuteron wavefunction which has simple analytic properties, and which closely approximates the square well solution is the so-called Hulthén wavefunction[29], viz.

$$f(r) = C(e^{-r/\alpha} - e^{-r/\beta}) \qquad \text{(IV.48)}$$

where the parameters are experimentally determined. (IV.48) is a difference of exponentials, and its fourier transform would be a sensible choice for $\psi_b^{(o)}(k)$, the trial wave function in (IV.47). There is a simpler solution, viz. the solution to a square well potential which is a single exponential[28]. The increase in the degree of difficulty of computation of (IV.47) when one goes from a single exponential to a function like (IV.48) is far too great to justify use of the Hulthén solution.

For an exponential of range $\frac{1}{\alpha}$ in configuration space, $\psi_b^{(o)}(k)$ is given by

$$\psi_b^{(o)}(k) = \sqrt{\frac{32\alpha^5}{\pi}} \frac{1}{(k^2+\alpha^2)^2} \qquad \text{(IV.49)}$$

$$= \frac{A}{(k^2+\alpha^2)^2}$$

With this, (IV.47) may be written as

$$ME = -A^2 \left(\int_0^\infty \frac{k^4 dk}{(k^2+\alpha^2)^4} + \int_0^\infty \frac{k^2 F_b(k)}{(k^2+\alpha^2)^4}\, dk \right.$$

$$+ \frac{Mg^2}{2(2\pi)^2} \int_0^\infty dk \int_0^\infty dq\ \ln \left| \frac{(q^2+k^2+2ME)^2 - 4M^2(q-k)^2 - 4M^2m^2}{(q^2+k^2+2ME)^2 - 4M^2(q+k)^2 - 4M^2m^2} \right| \times$$

$$\left. \times \frac{k}{(k^2+\alpha^2)^2} \frac{q}{(q^2+\alpha^2)^2} \right) . \qquad \text{(IV.50)}$$

The three terms can be identified respectively as corresponding to kinetic energy (k), self-energy (s) and potential energy (p), and therefore (IV.50) is rewritten as

$$ME = I_k + I_s + I_p$$

with

$$I_k = -A^2 \int_0^\infty \frac{k^4 dk}{(k^2+\alpha^2)^4} \qquad \text{(IV.51)}$$

$$I_s = -A^2 \int_0^\infty \frac{k^2 F_b(k)\, dk}{(k^2+\alpha^2)^4} \qquad \text{(IV.52)}$$

$$I_p = - \frac{Mg^2A^2}{2(2\pi)^2} \int_0^\infty dk \int_0^\infty dq \; \ell n \left| \frac{(q^2+k^2+2ME)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2+2ME)^2-4M^2(q+k)^2-4M^2m^2} \right| \times$$

$$\times \frac{k}{(k^2+\alpha^2)^2} \frac{q}{(q^2+\alpha^2)^2} \cdot \qquad \text{(IV.53)}$$

I_k can be evaluated exactly; in fact

$$I_k = - A^2 \cdot \frac{\pi}{32\alpha 3} = - \alpha^2$$

I_s and I_p, on the other hand, can be done analytically up to a stage; beyond that, the computation has to be numerically performed.

The calculation performed is the following: for a given value of g^2, the expression (IV.50) is computed over a range of values of α; the maximum value of E, and its location in α are then obtained. This is repeated for different values of the coupling constant, until the best pair of values of α and g that gives an acceptable value of E is obtained. (α was restricted to values below the one pion mass). It turns out that the computation is very sensitive to changes in the coupling constant, in the sense that the maximum changes value and location quite sharply with changes in g. The set of values that has been selected was at $g^2 = 4.6$, $\alpha = 122$ MeV, for which the maximum is 2.206 MeV. When g is increased, the maximum increases

and moves to a larger value of α. Whereas, for the square well, the range of the exponential is $\frac{1}{\sqrt{ME}}$ ($\sqrt{ME}$ = 45.69 MeV), the range one gets here is more of the order of one pion mass. Tables I - III contain the results of the calculation for three values of g^2. Appendix E outlines the procedure of integrating I_s and I_p.

IV.7 Evaluation of Phase Shifts

The phase shifts for the ℓth partial wave are given by the corresponding on-shell T-matrix via the relation

$$T_\ell(p) = -\frac{2}{\pi pM} e^{i\delta_\ell(p)} \sin\delta_\ell(p) \qquad \text{(IV.54)}$$

Here, as in subsequent sections, the on-shell T-matrix will be labelled by the on-shell momentum i.e. $T(\vec{p},\vec{p}) \equiv T(\vec{p})$. To obtain the phase shifts, therefore, one has to solve the integral equation (IV.33), and then go on to the mass-shell. It is quite obvious that an analytic solution to (IV.33), though desirable, is certainly not easily obtained. Therefore, the solution is obtained numerically using a technique of matrix inversion.

For s-waves, we first rewrite (IV.33) as follows:

$$T_o(k,p) = A(k,p) + B(k,p)\int_0^\infty \frac{q^2 dq}{p^2-q^2+i\eta} V_p(k,q)T_o(q,p) \qquad \text{(IV.55)}$$

Table I

Binding energy calculation (for $g^2 = 4.25$)

α (MeV)	$\frac{1}{Mg^2} I_p$	$\frac{1}{Mg^2} I_s$	E (MeV)
70	0.1550×10^{-2}	-0.1139×10^{-3}	0.5046
80	0.2024×10^{-2}	-0.1361×10^{-3}	0.7094
90	0.2546×10^{-2}	-0.1628×10^{-3}	0.8724
100	0.3112×10^{-2}	-0.1960×10^{-3}	0.9699
110	0.3717×10^{-2}	-0.2383×10^{-3}	0.9767
120	0.4359×10^{-2}	-0.2932×10^{-3}	0.8650

Table II

Binding energy calculation (for $g^2 = 4.60$)

α (MeV)	$\frac{1}{Mg^2} I_p$	$\frac{1}{Mg^2} I_s$	E (MeV)
118	0.4227×10^{-2}	-0.2804×10^{-3}	2.1961
119	0.4293×10^{-2}	-0.2879×10^{-3}	2.1934
120	0.4359×10^{-2}	-0.2932×10^{-3}	2.2001
121	0.4425×10^{-2}	-0.2983×10^{-3}	2.2055
122	0.4491×10^{-2}	-0.3058×10^{-3}	2.2009
123	0.4557×10^{-2}	-0.3123×10^{-3}	2.1990

Table III

Binding energy calculation (for $g^2 = 5.00$).

α (MeV)	$\frac{1}{Mg^2} I_p$	$\frac{1}{Mg^2} I_s$	E (MeV)
70	0.1550×10^{-2}	-0.1139×10^{-3}	1.5152
80	0.2024×10^{-2}	-0.1361×10^{-3}	2.0383
90	0.2546×10^{-2}	-0.1628×10^{-3}	2.5497
100	0.3112×10^{-2}	-0.1960×10^{-3}	3.0218
110	0.3717×10^{-2}	-0.2383×10^{-3}	3.4247
120	0.4359×10^{-2}	-0.2932×10^{-3}	3.7259

with

$$A(k,p) = \frac{g^2}{(2\pi)^3} \; 2\, M(p,k) V_p(k,p) \qquad \text{(IV.56)}$$

$$B(k,p) = \frac{g^2}{(2\pi)^3} \; 2\, M(p,k)\, M \qquad \text{(IV.57)}$$

$$V_p(k,q) = \frac{1}{2kq} \ln \left| \frac{(q^2+k^2-2p^2)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2-2p^2)^2-4M^2(q+k)^2-4M^2m^2} \right| \qquad \text{(IV.58)}$$

Equation (IV.55) therefore becomes

$$T_o(k,p) + B(k,p) \int_0^\infty dq \, \frac{q^2 V_p(k,q)}{p^2-q^2+i\eta} \, T_o(q,p) = A(k,p) \qquad \text{(IV.59)}$$

which is then written in the form of a matrix equation. p is kept as a parameter in the calculation, and for a given value of k, $T_o(k,p)$ is obtained for various values of k. In other words, half-off-shell matrix elements are obtained by this technique. The details of the computation are contained in Appendix D.

The real and imaginary parts of $T_o(k,p)$ are printed out, and the corresponding phase shift $\delta_o(p)$ (both real and imaginary parts) as well, utilising eqn. (IV.54). In all the computations done, the imaginary part of the phase shift is at most of the order of 10^{-13} radians. This is to be

expected, of course, as one is dealing with effects below the one-pion threshold i.e. no absorptive or dissipative effects enter. Also, for the same reason, unitarity can also be shown to hold almost exactly: the potential is real, and no approximations made should affect unitarity. In fact, the half-off-shell unitarity condition reads

$$\text{Im } T_o(k,p) = -\frac{\pi p}{2} T_o(k,p) T_o^*(p)$$

and the data show that for any value of p and k, this is almost exactly valid. The difference between the two sides of the relation is of the order of 10^{-6}; this arises from the numerical (as opposed to the exact and analytical) approach, and is, for practical purposes, equal to zero.

Three factors have to be considered in these calculations. The computations can be done for various values of the coupling constant; in addition, sensitivity to the cut-off and to the self-energy can also be tested. The cut-off, if present, restricts the range of the q-integration; ideally, one would like to see no variation of the results with a variation in the cut-off. The larger the cut-off, one would expect more structure introduced into the integral in eqn. (IV.59); hence if the structure adversely affects the numerics, lowering the cut-off should reduce the structure, thereby improving the numerical work.

As for the self-energy, the matrix inversion can be performed both in the presence and the absence of the self-energy term.

The most interesting results are shown in Graph I. This contains 3 curves, all with the coupling constant

$$\frac{g^2}{(2\pi)^3} = 0.0185$$

The three curves cover three possible situations with a fixed coupling constant, viz.

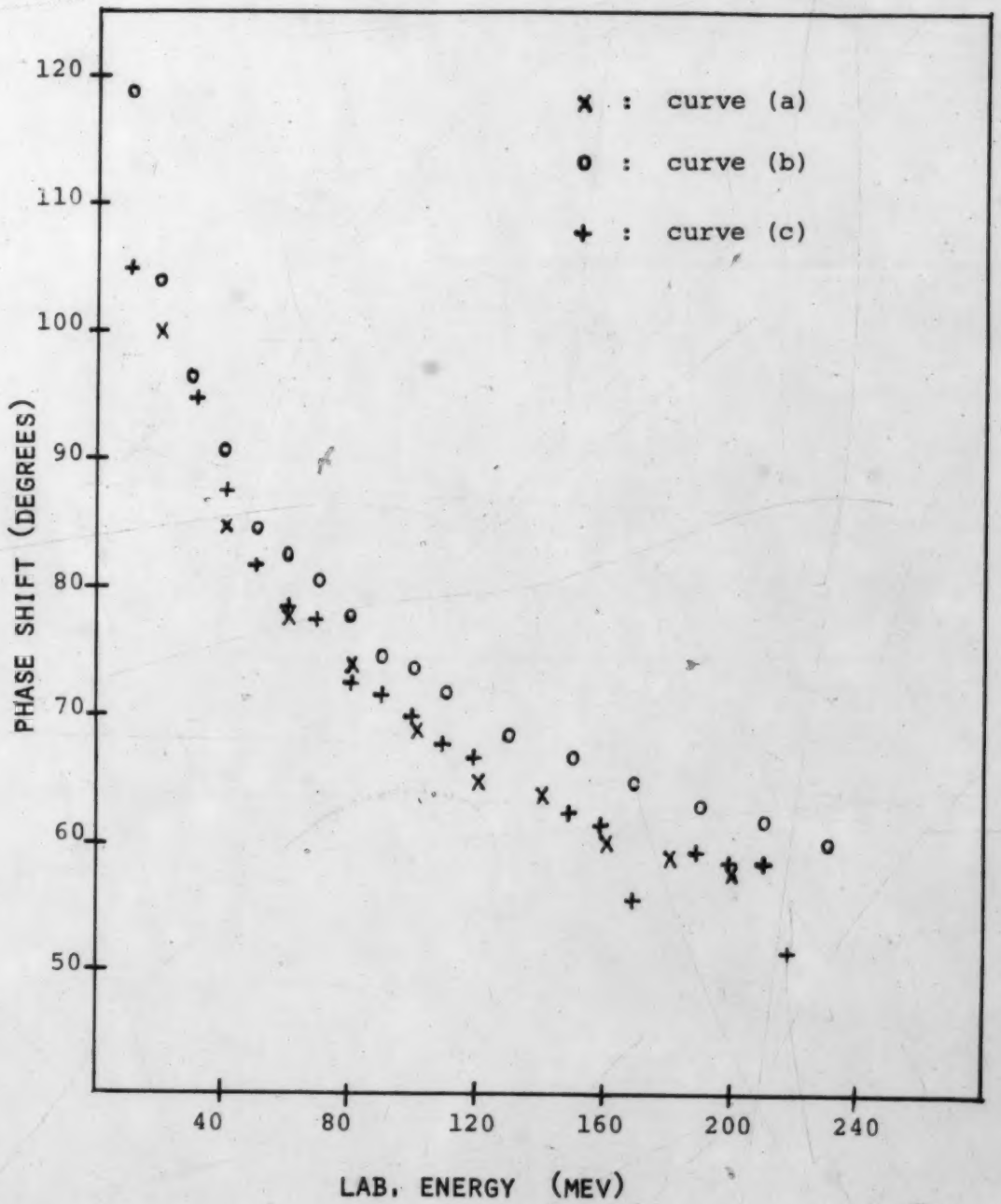
a) Excluding self-energy; cut-off = 1 nucleon mass;
b) Excluding self-energy; removing cut-off;
c) Including self-energy; removing cut-off.

It is obvious that in going from a) to c), more effects are being incorporated into the theory. As the curves indicate, the agreement between these three cases is reasonably good. The phase shift seems to drop from a value of 180°, therefore indicating the existence of a bound state.

Further discussion of the results appears in Chapter VI.

FIGURE I

SCALAR MODEL: PHASE SHIFTS



V. THE PSEUDOSCALAR MODEL

In reality, the proton and the neutron have different charges - they form an isospin doublet; in addition, they are spin-½ particles. As opposed to the scalar theory, these features are incorporated into the pseudoscalar model. Consequently the states can no longer be simply defined by momenta and binding energies; spins and isospins have to be included, too. With all the indices suppressed for the moment, the equations of motion are:

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M}\right)\psi(x_o,\vec{x}) = -f_o\vec{\tau}\cdot\sigma^j\psi(x_o,\vec{x})\partial^j\vec{\phi}(x_o,\vec{x}) \qquad (V.1)$$

$$(\Box + m^2)\vec{\phi}(x_o,\vec{x}) = -f_o\partial^i\psi^\dagger(x_o,\vec{x})\sigma^i\vec{\tau}\psi(x_o,\vec{x}) \qquad (V.2)$$

Here, ψ, $\vec{\phi}$ are the nucleon and pion fields respectively, $\vec{\tau}$ and $\vec{\sigma}$ are the usual isospin and spin matrices, and f_o is the pseudoscalar coupling constant. (The time variable is x_o, as opposed to t, which has been used until now, and which will now be reserved for isospin.) Eqns. (V.1) and (V.2) are analogous to (IV.1) and (IV.2), and, as before, because the interaction term in (V.2) is $\vec{\phi}$-independent, it can be solved for $\vec{\phi}(x_o,\vec{x})$ and inserted into (V.1) to give the following equation, in all its glory, for the nucleon field:

$$(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M})\psi_{\dot{\alpha}\alpha'}(x_o,\vec{x}) = f_o^2\sigma_{\alpha\beta}\vec{\tau}_{\alpha'\beta'}\psi_{\beta\beta'}(x_o,\vec{x})\partial^i\partial^k \int d^3x'dx'_o$$

$$\vec{\Delta}(x_o-x'_o;\vec{x}-\vec{x}')\psi^{\dagger}_{\gamma\gamma'}(x'_o,\vec{x}')\vec{\tau}_{\gamma'\delta'}\sigma^k_{\gamma\delta}\psi_{\delta\delta'}(x'_o,\vec{x}') \qquad (V.3)$$

(As before, we have no free pion field, and hence only the principal value Green's function.) In (V.3), primed (unprimed) suffixes refer to isospin (spin) indices; there is summation over repeated indices. Apart from the complications of spin and isospin, (V.3) differs from (IV.4) in that it contains the double derivative $\partial^i\partial^k$. The field operator ψ is, of course, a matrix in spin-isospin space, as evidenced by the suffixes.

As already mentioned, the physical states of this theory are no longer defined as before. The vacuum $|0\rangle$ stays unchanged; one particle states are defined by momentum, spin and isospin, and their third components. Therefore, in full, one should write such states as $|\vec{p}STs_3t_3\rangle$, S(T) being the spin (isospin), and s_3 and t_3 the corresponding third components. However, as $S = T = \frac{1}{2}$ in this case, they will therefore be suppressed; instead, the third components only will be specified, and will be relabelled s,t. So the one-particle states are $|\vec{p}st\rangle$. Two particle scattering states are, in analogous fashion, $|\vec{p}_1\vec{p}_2STs_3t_3\rangle$, where now S(T) labels total spin (isospin), and s_3 and t_3 are their respective third components. Assuming that there is only

one bound state (the deuteron), it is given by $|pE_bSTs_3t_3>$. For discussion of the NN problem, this is as far as we need to go.

Analogous to (III.2) and (III.3), the commutators read:

$$<0|[\psi_{\alpha\alpha'}(x_o,\vec{x}),\psi^{\dagger}_{\beta\beta'}(x_o,\vec{x}')]_{\pm}|0> = \delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta(\vec{x}-\vec{x}') \quad (V.4)$$

$$<\vec{p}st|[\psi_{\alpha\alpha'}(x_o,\vec{x}),\psi^{\dagger}_{\beta\beta'}(x_o,\vec{x}']_{\pm}|\vec{p}'s't'> =$$

$$\delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{ss'}\delta_{tt'}\delta(\vec{x}-\vec{x}')\delta(\vec{p}-\vec{p}'). \quad (V.5)$$

With eqns. (V.3), (V.4) and (V.5), one has all the necessary equations.

V.1 One Nucleon Equation

For the one-particle wavefunction, one tries the following Ansatz:

$$<0|\psi_{\alpha\alpha'}(x_o,\vec{x})|\vec{p}st> = \delta_{\alpha s}\delta_{x't}u_p(x_o,\vec{x}) \quad (V.6)$$

$u_p(x_o,\vec{x})$ should, in fact, be labelled by the total spin and isospin, which in this case, as mentioned before, can only be ½, and are therefore suppressed. (The spatial part of the wavefunction is labelled only by total spin and isospin.) Once again, one assumes free solutions (which form

a complete set) i.e.

$$u_p(x_o,\vec{x}) = Ae^{-iE_p x_o} e^{i\vec{p}\cdot\vec{x}} \tag{V.7}$$

Application of (V.4) to (V.6) gives the value of A; i.e. one gets that

$$\langle 0|\psi_{\alpha\alpha'}(x_o,\vec{x})|\vec{p}st\rangle = \frac{1}{(2\pi)^{3/2}} \delta_{\alpha s}\delta_{\alpha' t} e^{-iE_p x_o} e^{i\vec{p}\cdot\vec{x}} \tag{V.8}$$

The equation for the one particle wavefunction is derived as before (App. F) and gives the following equation for $u_p(x_o,\vec{x})$:

$$(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M}) u_p(x_o,\vec{x}) =$$

$$3f_o^2 \int dx_o' \, d^3x' \, d^3p' \; u_{p'}(x_o,\vec{x}) \nabla_x^2 \bar{\Delta}(x_o - x_o'; \vec{x}-\vec{x}') u_{p'}^*(x_o',\vec{x}') u_p(x_o',\vec{x}') \tag{V.9}$$

As was the case in the scalar model, this non-linear equation for u_p leads to an integral equation for the energy, i.e.

$$E_p = \frac{p^2}{2M} + 3(2\pi)W_p \tag{V.10}$$

which is eqn. (F5). In obtaining (V.10), quantities W_p and $\tilde{\tilde{W}}$ [analogous to V_p (III.8) and $\tilde{\tilde{V}}$ (III.9)] are defined:

$$W_p = \int d^3q \, \tilde{\tilde{W}}[E_q - E_p; \vec{q}-\vec{p}] \tag{V.11}$$

$$\tilde{\tilde{W}}(\omega,\vec{k}) = -\frac{f_o^2}{(2\pi)^4} k^2 \int dy_o \, d^3y \, \bar{\Delta}(y_o,\vec{y}) e^{-i\omega y_o} e^{i\vec{k}\cdot\vec{y}}$$

$$= \frac{f_o^2}{(2\pi)^4} P \frac{k^2}{\omega^2 - k^2 - m^2} \tag{V.12}$$

In addition to these, one defines, for future use, a quantity $W_o(\vec{x})$ [cf. $V_o(\vec{x})$ (III.19)] as follows

$$W_o(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3p \, e^{-i\vec{p}\cdot\vec{x}} W_p \tag{V.13}$$

So, as before, the true energy is the solution to (V.10). However, in practice, a closed solution cannot be obtained. As for the previous model, therefore, the self-energy will be dropped wherever E_p appears in the kernel for the two-particle equation.

V.2 The Two-Nucleon System

A typical 2-particle wavefunction looks like

$$\langle 0 | \psi_{\alpha\alpha'}(x_o,\vec{z}) \psi_{\beta\beta'}(x_o,\vec{x}) | \vec{p}_1 \vec{p}_2 S T s_3 t_3 \rangle \tag{V.14}$$

The equation governing the behaviour of objects such as (V.14) is derived exactly as outlined in Section III, once the spin-isospin dependence is properly accounted for. This is done as follows:

$$
\begin{aligned}
&\langle 0|\psi_{\alpha\alpha'}(x_o,\vec{z})\chi_{\beta\beta'}(x_o,\vec{x})|\vec{p}_1\vec{p}_2 STs_3t_3\rangle \\
&= \sum_{st}\int d^3q \langle 0|\psi_{\alpha\alpha'}(x_o,\vec{z})|\vec{q}st\rangle\langle\vec{q}st|\psi_{\beta\beta'}(x_o,\vec{x})|\vec{p}_1\vec{p}_2 STs_3t_3\rangle \\
&= \int d^3q\, u_q(x_o,\vec{z})\langle\vec{q}\alpha\alpha'|\psi_{\beta\beta'}(x_o,\vec{x})|\vec{p}_1\vec{p}_2 STs_3t_3\rangle \\
&= \langle\alpha\beta|Ss_3\rangle\langle\alpha'\beta'|Tt_3\rangle F^{ST}_{p_1p_2}(x_o,\vec{x},\vec{z}) \qquad \text{(V.15)}
\end{aligned}
$$

which serves to define the wavefunction of the two-particle system, viz. $F^{ST}_{p_1p_2}(x_o,\vec{x},\vec{z})$. A similar relationship exists for the bound state. One can then derive equations for $F^{ST}_{p_1p_2}(x_o,\vec{x},\vec{z})$ and $F^{ST}_{Ep}(x_o,\vec{x},\vec{z})$ as before (see App. G). Eqn. (G8) is the scattering state wavefunction when S(T) is the total spin (isospin). Because of the different products of spin and isospin wavefunctions (e.g. $\langle s_3'\nu|Ss_3\rangle\langle t_3'\nu'|Tt_3\rangle$), it is not apparent whether or not S' and T' (the indices for the intermediate states) can take all values. Rewriting (G8) in full gives:

$$\langle s_3'\nu|Ss_3\rangle\langle t'\nu'|Tt_3\rangle\left(i\frac{\partial}{\partial x_o}+\frac{\nabla_x^2+\nabla_z^2}{2M}\right)F^{ST}_{p_1p_2}(x_o,\vec{z},\vec{x})$$

$$=3(2\pi)\langle s_3'\nu|Ss_3\rangle\langle t_3'\nu'|Tt_3\rangle\int d^3x'\ W_o(\vec{x}-\vec{x}')F^{ST}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+f_o^2\,\sigma^i_{\nu\beta}\vec{\tau}_{\nu'\beta'}\sigma^k_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\int\langle s_3'\beta|S's_3''\rangle\langle t_3'\beta'|T't_3''\rangle\langle S's_3''|s_3'''\gamma\rangle\times$$

$$\times\langle T't_3''|t_3'''\gamma'\rangle\langle s_3'''\delta|Ss_3\rangle\langle t_3'''\delta'|Tt_3\rangle F^{S'T'}_{k_1k_2}(x_o,\vec{z},\vec{x})\times$$

$$\times\partial^z_k\partial^{\bar{z}}_i\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')F^{S'T'*}_{k_1k_2}(x_o',\vec{x}',\vec{z})\times$$

$$\times F^{ST}_{p_1p_2}(x_o',\vec{x}',\vec{z}')\,d^3x'\,dx_o'\,d^3k_1\,d^3k_2\,d^3z'$$

$$+f_o^2\,\sigma^i_{\nu\beta}\vec{\tau}_{\nu'\beta'}\sigma^k_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\int\langle s_3'\beta|S's_3''\rangle\langle t_3'\beta'|T't_3''\rangle\langle S's_3''|s_3'''\gamma\rangle\times$$

$$\times\langle T't_3''|t_3'''\gamma'\rangle\langle s_3'''\delta|Ss_3\rangle\langle t_3'''\delta'|Tt_3\rangle F^{S'T'}_{k_1E}(x_o,\vec{z},\vec{x})\times$$

$$\times\partial^z_k\partial^{\bar{z}}_i\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')F^{S'T'*}_{k_1E}(x_o',\vec{x}',\vec{z}')\times$$

$$\times F^{ST}_{p_1p_2}(x_o',\vec{x}',\vec{z}')\,d^3x'\,dx_o'\,d^3k_1\,d^3z'\ .$$

(V.16)

It is obvious from (V.16) that each state labelled by (S,T) couples to a number of possible states (S',T'). It will be only by looking at each pair (S,T) that one can determine which values (S',T') actually do appear. Because the two particle states have both spin and isospin equal to ½, S and T can only take the values 0 and 1. There is

fairly lengthy algebra involved in actually obtaining the separate equations for each (S,T) pair, and this removal of the Clebsch-Gordan coefficients is done in Appendices H and I respectively.

Since the deuteron bound state is an S = 1, T = 0 state, and in fact, the only bound state of the system, certain intermediate state contributions to (V.16) will drop out, i.e. the only bound state wavefunction is F^{10}_{kE}, and this will simplify some of the equations.

Taking account of this, and using expressions (H9), (H10), (I6) and (I7), the following equations are obtained for the four scattering channels and the single bound state.

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right) F^{00}_{p_1p_2}(x_o,\vec{z},\vec{x}) = 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}') F^{00}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+ 3f_o^2 \int F^{11}_{k_1k_2}(x_o,\vec{z},\vec{x})\nabla^2\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}') F^{11*}_{k_1k_2}(x_o',\vec{x}',\vec{z}') \times$$

$$\times F^{00}_{p_1p_2}(x_o',\vec{x}',\vec{z}')\, dx_o'\, d^3x'\, d^3z'\, d^3k_1\, d^3k_2 \qquad \text{(V.17)}$$

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right) F^{01}_{p_1p_2}(x_o,\vec{z},\vec{x}) = 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}') F^{01}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+ f_o^2 \int F^{10}_{k_1k_2}(x_o,\vec{z},\vec{x})\nabla^2\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}') F^{10*}_{k_1k_2}(x_o',\vec{x}',\vec{z}') \times$$

$$\times F^{01}_{p_1p_2}(x_o',\vec{x}',\vec{z}')\, d^3x'\, dx_o'\, d^3z'\, d^3k_1\, d^3k_2$$

$$+ 2f_o^2 \int F^{11}_{k_1k_2}(x_o,\vec{z},\vec{x})\nabla_z^2\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')F^{11*}_{k_1k_2}(x_o',\vec{x}',\vec{z}') \times$$

$$\times F^{01}_{p_1p_2}(x_o',\vec{x}',\vec{z}')\, d^3x'\, dx_o'\, d^3z'\, d^3k_1\, d^3k_2$$

$$+ f_o^2 \int F^{10}_{kE}(x_o,\vec{z},\vec{x})\nabla_z^2\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')F^{10*}_{kE}(x_o',\vec{x}',\vec{z}') \times$$

$$\times F^{01}_{p_1p_2}(x_o',\vec{x}',\vec{z}')\, d^3x'\, dx_o'\, d^3z'\, d^3k \quad . \qquad (V.18)$$

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right)F^{10}_{p_1p_2}(x_o,\vec{z},\vec{x}) = 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}')F^{10}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+ \frac{3}{2}f_o^2 \int F^{01}_{k_1k_2}(x_o,\vec{z},\vec{x})\,[\nabla_z^2 - (\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';z-x') \times$$

$$\times F^{01*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{10}_{p_1p_2}(x_o',\vec{x}',\vec{z}')dx_o'd^3x'd^3z'd^3k_1d^3k_2$$

$$+ \frac{3}{2}f_o^2 \int F^{11}_{k_1k_2}(x_o,\vec{z},\vec{x})\,[\nabla_z^2 + (\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}') \times$$

$$\times F^{11*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{10}_{p_1p_2}(x_o',\vec{x}',\vec{z}')dx_o'd^3x'd^3z'd^3k_1d^3k_2 \quad .$$

$$(V.19)$$

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right)F^{11}_{p_1p_2}(x_o,\vec{z},\vec{x}) = 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}')F^{11}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+ \frac{1}{2}f_o^2 \int F^{00}_{k_1k_2}(x_o,\vec{z},\vec{x})\,[\nabla_z^2 - (\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}') \times$$

$$\times F^{00*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{11}_{p_1p_2}(x_o',\vec{x}',\vec{z}')d^3x'dx_o'd^3z'd^3k_1d^3k_2$$

$$+\tfrac{1}{2}f_o^2\int F^{10}_{k_1k_2}(x_o,\vec{z},\vec{x})[\nabla_z^2+(\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\times$$

$$\times F^{10*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{11}_{p_1p_2}(x_o',\vec{x}',\vec{z}')d^3x'dx_o'd^3z'd^3k_1d^3k_2$$

$$+f_o^2\int F^{01}_{k_1k_2}(x_o,\vec{z},\vec{x})[\nabla_z^2-(\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\times$$

$$\times F^{01*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{11}_{p_1p_2}(x_o',\vec{x}',\vec{z}')d^3x'dx_o'd^3z'd^3k_1d^3k_2$$

$$+f_o^2\int F^{11}_{k_1k_2}(x_o,\vec{z},\vec{x})[\nabla_z^2+(\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\times$$

$$\times F^{11*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{11}_{p_1p_2}(x_o',\vec{x}',\vec{z}')d^3x'dx_o'd^3z'd^3k_1d^3k_2$$

$$+\tfrac{1}{2}f_2^o\int F^{10}_{kE}(x_o,\vec{z},\vec{x})[\nabla_z^2+(\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\times$$

$$\times F^{10*}_{kE}(x_o',\vec{x}',\vec{z}')F^{11}_{p_1p_2}(x_o',\vec{x}',\vec{z}')d^3x'dx_o'd^3z'd^3k\ .$$

(V.20)

$$\left(i\frac{\partial}{\partial x_o}+\frac{\nabla_x^2+\nabla_z^2}{2M}\right)F^{10}_{pE}(x_o,\vec{z},\vec{x})=3(2\pi)\int d^3x'W_o(\vec{x}-\vec{x}')F^{10}_{pE}(x_o,\vec{z},\vec{x}')$$

$$+\tfrac{3}{2}f_o^2\int F^{01}_{k_1k_2}(x_o,\vec{z},\vec{x})[\nabla_z^2-(\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)]\times$$

$$\times F^{01*}_{k_1 k_2}(x'_o,\vec{x}',\vec{z}')F^{10}_{pE}(x'_o,\vec{x}',\vec{z}')dx'_o d^3x' d^3z' d^3k_1 d^3k_2$$

$$+ \tfrac{3}{2}f_o^2 \int F^{11}_{k_1 k_2}(x_o,\vec{z},\vec{x})[\nabla_z^2 + (\vec{\sigma}_1\cdot\vec{\nabla}_z)(\vec{\sigma}_2\cdot\vec{\nabla}_z)] \times$$

$$\times F^{11*}_{k_1 k_2}(x'_o,\vec{x}',\vec{z}')F^{10}_{pE}(x'_o,\vec{x}',\vec{z}')dx'_o d^3x' d^3z' d^3k_1 d^3k_2 \ .$$

(V.21)

Eqns. (V.17) to (V.21) form the set of coupled, non-linear, and non-local equations which describes the 2-nucleon system in the framework of this theory. There is really no hope of being able to decouple these equations; therefore, they will be linearised as was done in the scalar model. It is also apparent that the non-local "potentials" which appear in these equations differ in all the equations - hence each state has an inherently different potential, each of which is both non-local and energy-dependent. It can also be seen that the presence of $(\vec{\sigma}_1\cdot\vec{\nabla})(\vec{\sigma}_2\cdot\vec{\nabla})$ terms leads to tensor terms in some of the channels; in particular, in the bound state equation, where in fact, such a term is expected to be present. Also in the bound state equation, viz. (V.21) the intermediate states do not include the bound state itself. (This has already been mentioned in Section IV.1.)

As described in Section III (viz. eqns. (III.12) to (III.16)), these functions F^{ST} can be replaced by the centre-of-mass wavefunctions f^{ST}, and equations (V.17) to (V.21), which are actually two-particle Schrödinger equations, can be reduced to single particle equations in the relative coordinate. The actual derivation will <u>not</u> be done here, as the identical computation is to be found in ref. 22. In brief, however, it involves the following: first, consider the derivatives that appear within the kernel.

$$\nabla_z^2 \bar{\Delta}(x_o - x_o'; \vec{z} - \vec{x}') = \frac{1}{(2\pi)^4} \int k^2 \frac{e^{i\omega(x_o - x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')}}{\omega^2 - k^2 - m^2} d\omega d^3k$$

$$= \frac{1}{f_o^2} \int \tilde{\tilde{W}}(\omega, \vec{k}) e^{i\omega(x_o - x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')} d\omega d^3k \ . \qquad (V.22)$$

$$(\vec{\sigma}_1 \cdot \vec{\nabla}_z)(\vec{\sigma}_2 \cdot \vec{\nabla}_z) \bar{\Delta}(x_o - x_o'; \vec{z} - \vec{x}') =$$

$$= \frac{1}{(2\pi)^4} \int \frac{(\vec{\sigma}_1 \cdot \vec{k})(\vec{\sigma}_2 \cdot \vec{k}) e^{i\omega(x_o - x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')}}{\omega^2 - k^2 - m^2} d\omega d^3k$$

$$= \frac{1}{f_o^2} \int \tilde{\tilde{W}}_1(\omega, \vec{k}) e^{i\omega(x_o - x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')} d\omega d^3k \ . \qquad (V.23)$$

with $\tilde{\tilde{W}}(\omega,\vec{k})$ defined by (V.12) and

$$\tilde{\tilde{W}}_1(\omega,\vec{k}) = \frac{f_o^2}{(2\pi)^4} P \frac{(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})}{\omega^2-k^2-m^2} \qquad (V.24)$$

The equations that will be examined in closer detail are the scattering equation for the (0,1) channel, (V.17), and the deuteron equation, (V.21). Using eqns. (V.22) and (V.23), these may be rewritten as follows:

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right) F^{01}_{p_1p_2}(x_o,\vec{z},\vec{x}) = 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}') F^{01}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+ \int [F^{10}_{k_1k_2}(x_o,\vec{z},\vec{x}) F^{10*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')$$

$$+ 2F^{11}_{k_1k_2}(x_o,\vec{z},\vec{x}) F^{11*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')] F^{01}_{p_1p_2}(x_o',\vec{x}',\vec{z}') \times$$

$$\times \tilde{\tilde{W}}(\omega,\vec{k}) e^{i\omega(x_o-x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')} d\omega d^3k dx_o' d^3x' d^3z' d^3k_1 d^3k_2$$

$$+ \int F^{10}_{k_1E}(x_o,\vec{z},\vec{x}) F^{10*}_{k_1E}(x_o',\vec{x}',\vec{z}') F^{01}_{p_1p_2}(x_o',\vec{x}',\vec{z}') \times$$

$$\times \tilde{\tilde{W}}(\omega,\vec{k}) e^{i\omega(x_o-x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')} d\omega d^3k dx_o' d^3x' d^3z' d^3k_1 \,. \qquad (V.25)$$

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right) F_{pE}^{10}(x_o,\vec{z},\vec{x}) = 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}') F_{pE}^{10}(x_o,\vec{z},\vec{x}')$$

$$+ \frac{3}{2}\int [F_{k_1k_2}^{01}(x_o,\vec{z},\vec{x}) F_{k_1k_2}^{01*}(x_o',\vec{x}',\vec{z}') + F_{k_1k_2}^{11}(x_o,\vec{z},\vec{x}) F_{k_1k_2}^{11*}(x_o',\vec{x}',\vec{z}')] \times$$

$$\times \tilde{\tilde{W}}(\omega,\vec{k}) e^{i\omega(x_o-x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')} F_{pE}^{10}(x_o',\vec{x}',\vec{z}') d\omega d^3k dx_o' d^3x' d^3z' d^3k_1 d^3k_2$$

$$- \frac{3}{2}\int [F_{k_1k_2}^{01}(x_o,\vec{z},\vec{x}) F_{k_1k_2}^{01*}(x_o',\vec{x}',\vec{z}') - F_{k_1k_2}^{11}(x_o,\vec{z},\vec{x}) F_{k_1k_2}^{11*}(x_o',\vec{x}',\vec{z}')] \times$$

$$\times \tilde{\tilde{W}}(\omega,\vec{k}) e^{i\omega(x_o-x_o')} e^{-i\vec{k}\cdot(\vec{z}-\vec{x}')} F_{pE}^{10}(x_o',\vec{x}',\vec{z}') d\omega d^3k dx_o' d^3x' d^3z' d^3k_1 d^3k_2. \qquad (V.26)$$

The corresponding equations for the other channels can also be brought to this form. Eventually, after defining the centre-of-mass wavefunctions (in analogy with eqns. (III.12) to (III.16)), and after a certain amount of algebra (identical to that done in ref. 22), one is led to the following set of equations:

$$(\nabla_r^2 + p^2) f_p^{01}(\vec{r}) = 3(2\pi M) \int d^3r' \; W_o(\vec{r}-\vec{r}') f_p^{01}(\vec{r}')$$

$$+ M\int d^3r' \int d^3q W_{pq}(\vec{r}-\vec{r}') [f_q^{10}(\vec{r}) f_q^{10*}(\vec{r}') + 2 f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_p^{01}(\vec{r}')$$

$$+ M \int d^3r' \; W_{pE}(\vec{r}-\vec{r}') f_b^{10}(\vec{r}) f_b^{10}(\vec{r}') f_p^{01}(\vec{r}') \qquad (V.27)$$

$$(\nabla_r^2 + p^2) f_p^{00}(\vec{r}) = 3(2\pi M) \int d^3r'\ W_o(\vec{r}-\vec{r}') f_p^{00}(\vec{r}')$$

$$+ 3M \int d^3r' \int d^3q\ W_{pq}(\vec{r}-\vec{r}') f^{11}_{k_1k_2}(\vec{r}) f^{11*}_{k_1k_2}(\vec{r}') f_p^{00}(\vec{r}') \quad \text{(V.28)}$$

$$(\nabla_r^2 + p^2) f_p^{10}(\vec{r}) = 3(2\pi M) \int d^3r'\ W_o(\vec{r}-\vec{r}') f_p^{10}(\vec{r}')$$

$$+ \tfrac{3}{2}M\int d^3r' \int d^3q W_{pq}(\vec{r}-\vec{r}') [f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') + f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_p^{10}(\vec{r}')$$

$$-\tfrac{3}{2}M\int d^3r' \int d^3q W^{(1)}_{pq}(\vec{r}-\vec{r}') [f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') - f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_p^{10}(\vec{r}')$$

(V.29)

$$(\nabla_r^2 + p^2) f_p^{11}(\vec{r}) = 3(2\pi M) \int d^3r'\ W_o(\vec{r}-\vec{r}') f_p^{11}(\vec{r}')$$

$$+ \tfrac{1}{2}M \int d^3r' \int d^3q\ W_{pq}(\vec{r}-\vec{r}') [f_q^{00}(\vec{r}) f_q^{00*}(\vec{r}') + f_q^{10}(\vec{r}) f^{10*}(\vec{r}')$$

$$+ 2\ f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') + 2 f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_p^{11}(\vec{r}')$$

$$- \tfrac{1}{2}M\int d^3r' \int d^3q W^{(1)}_{pq}(\vec{r}-\vec{r}') [f_q^{00}(\vec{r}) f_q^{00*}(\vec{r}') - f_q^{10}(\vec{r}) f_q^{10*}(\vec{r}')$$

$$+ 2\ f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') - 2 f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_p^{11}(\vec{r}')$$

$$+ \tfrac{1}{2}M\int d^3r' [W_{pE}(\vec{r}-\vec{r}') + W^{(1)}_{pE}(\vec{r}-\vec{r}')] f_b^{10}(\vec{r}) f_b^{10*}(\vec{r}') f_p^{11}(\vec{r}')$$

(V.30)

$$(\nabla_r^2 - ME) f_b^{10}(\vec{r}) = 3(2\pi M) \int d^3r'\, W_o(\vec{r}-\vec{r}') f_b^{10}(\vec{r}')$$

$$+ \frac{3}{2}M \int d^3r' \int d^3q W_{Eq}(\vec{r}-\vec{r}') [f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') + f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_b^{10}(\vec{r}')$$

$$- \frac{3}{2}M \int d^3r' \int d^3q W_{Eq}^{(1)}(\vec{r}-\vec{r}') [f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') - f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_b^{10}(\vec{r}')$$

(V.31)

In eqns. (V.27) to (V.31):

$$W_{pq}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \tilde{\tilde{W}}\left[\frac{q^2-p^2}{M} + \frac{k^2}{4M}\,;\, \vec{k}\right] \qquad (V.32)$$

$$W_{pE}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \tilde{\tilde{W}}\left[\frac{k^2-4p^2}{4M} - E_b;\, \vec{k}\right] \qquad (V.33)$$

$$W_{Eq}(\vec{x}) = (2\pi)^4 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \tilde{\tilde{W}}\left[\frac{q^2}{M} + \frac{k^2}{4M} + E_b;\, \vec{k}\right] \qquad (V.34)$$

$W_{pq}^{(1)}$, $W_{pE}^{(1)}$ and $W_{Eq}^{(1)}$ are obtained from these merely by subscripting the $\tilde{\tilde{W}}$'s on the r.h.s. (i.e. replacing them by $\tilde{\tilde{W}}_1$'s).

In eqns. (V.27) to (V.31), one has the non-local Schrödinger equations governing the bound state and the various scattering channels. These are the pseudoscalar analogues of (III.17) and (III.18). Similarly, the potentials defined in (V.32) to (V.34) are analogous to (III.20), and (III.22). Once again, therefore, one has obtained

differential equations for the various states, starting from a pair of coupled field equations, (V.1) and (V.2).

Most of our interest will focus on the scattering equation (V.27). It will be transformed into momentum space, and, as with the scalar model, a half-off-shell T-matrix will be constructed.

V.3 The S = 0, T = 1 Channel

Apart from the fact that this channel is simpler to handle, due to the absence of a tensor force, a great deal of information about neutron-proton scattering is contained in it. Therefore, we shall go through the calculations in some detail.

When the self-energy is subtracted out (as in the scalar model), the equation for scattering in this channel, viz. (V.27), becomes

$$(\nabla_r^2 + p^2) f_p^{01}(\vec{r}) = -3(2\pi M) \int d^3r'\, W_o(\vec{r}-\vec{r}') f_p^{01}(\vec{r}')$$

$$+ M\int d^3r' \int d^3q W_{pq}(\vec{r}-\vec{r}') [f_q^{10}(\vec{r}) f_q^{10*}(\vec{r}') + 2 f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] f_p^{01}(\vec{r}') \tag{V.35}$$

where, in addition, the bound state contribution has been dropped for the reasons already discussed in the scalar model. The steps involved in going from this, the config-

uration space equation, to the corresponding momentum space equation, are exactly identical to those involved in going from (IV.14) to (IV.26) in the scalar model. This gives

$$(p^2-k^2)\psi_p^{01}(\vec{k}) = -3(2\pi M)\int d^3q\ \tilde{\tilde{W}}\left[\frac{q^2-k^2}{2M}\ ;\ \vec{q}+\vec{k}\right]\psi_p^{01}(\vec{k})$$

$$+ 8M(2\pi)^7 \int d^3k_1\ d^3k_2\ d^3k_3 [\psi_{k_1}^{10(o)}(\vec{k}+\vec{k}_2-\vec{k}_3)\psi_{k_1}^{10(o)*}(\vec{k}_2)$$

$$+ 2\psi_{k_1}^{11(o)}(\vec{k}+\vec{k}_2-\vec{k}_3)\psi_{k_1}^{11(o)*}(\vec{k}_2)]\tilde{\tilde{W}}\left[\frac{(\vec{k}_2-\vec{k}_3)^2+k_1^2-p^2}{M};2(\vec{k}_3-\vec{k}_2)\right]\psi_p^{01}(\vec{k}_3)$$

(V.36)

In this case, however, there are two intermediate states to be considered, viz. the (1,0) and (1,1) states. As the former is a spin triplet, isospin singlet state, whereas the latter is both a spin and isospin triplet, the spatial parts of the wavefunctions differ. In the (1,0) case, it will be even, while in the (1,1) state it will be odd. And since these intermediate state wavefunctions are taken to be free, one has (see App. J)

$$\psi_q^{10(o)}(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[\delta(\vec{q}-\vec{k}) + \delta(\vec{q}+\vec{k})\right] \qquad (V.37)$$

$$\psi_q^{11(o)}(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[\delta(\vec{q}-\vec{k}) - \delta(\vec{q}+\vec{k})\right] \qquad (V.38)$$

This will lead to the following equation:

$$(p^2-k^2)\psi_p^{01}(\vec{k}) = -3(2\pi M)\int d^3q\ \tilde{\tilde{W}}\left[\frac{q^2-k^2}{2M}\ ;\ \vec{q}+\vec{k}\right]\psi_p^{01}(\vec{k})$$

$$+3(2\pi M)\int d^3q\ \tilde{\tilde{W}}\left[\frac{q^2+k^2-2p^2}{2M}\ ;\ \vec{q}+\vec{k}\right]\psi_p^{01}(\vec{k})$$

$$-2\pi M\int d^3q\ \tilde{\tilde{W}}\left[\frac{q^2+k^2-2p^2}{2M}\ ;\ \vec{q}+\vec{k}\right]\psi_p^{01}(\vec{q}) \qquad (V.39)$$

Eqn. (V.39) is identical in form to (IV.28); analogous to the F(p,k) of scalar theory, one can define

$$G(p,k) = 6\pi M\int d^3q\left\{\tilde{\tilde{W}}\left[\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right]-\tilde{\tilde{W}}\left[\frac{q^2-k^2}{2M};\vec{q}+\vec{k}\right]\right\} \qquad (V.40)$$

which reduces (V.39) to

$$[p^2-k^2-G(p,k)]\psi_p^{01}(\vec{k}) = -2\pi M\int d^3q\ \tilde{\tilde{W}}\left[\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right]\psi_p^{01}(\vec{q}) \qquad (V.41)$$

It is obvious from (V.40) that <u>on</u> the mass shell, p = k, G(p,k) vanishes. This has to be so, if the equation (V.41) is to be valid (see Section IV.5). Furthermore, a T-matrix can be constructed as before (eqn. IV.31), as the symmetry is the same as it was in the scalar model (S = 0,

T = 1 implies the spatial part of the wavefunction is even). Therefore,

$$\psi_p^{01}(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[\delta(\vec{p}+\vec{k})+\delta(\vec{p}-\vec{k})\right]+\frac{M}{\sqrt{2}(2\pi)^3}\frac{T^{01}(\vec{k},\vec{p},p^2)+T^{01}(-\vec{k},\vec{p},p^2)}{p^2-k^2+i\eta} \tag{V.42}$$

Substituting (V.42) into (V.41) gives the following equation for the T-matrix:

$$T^{01}(\vec{k},\vec{p}) = -\pi\frac{p^2-k^2}{p^2-k^2-G(p,k)}\left[\tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}-\vec{p}\right)+\tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}+\vec{p}\right)\right]$$

$$-M\frac{p^2-k^2}{p^2-k^2-G(p,k)}\int\frac{d^3q}{p^2-q^2+i\eta}\tilde{\tilde{W}}\left[\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right]T^{01}(\vec{q},\vec{p}) \tag{V.43}$$

Once again the third label, p^2, has been suppressed in T; the complete analogy between (V.43) and the corresponding scalar model equation, (IV.32), is very obvious.

The equation for the s-wave part of the T-matrix may be obtained by partial-wave decomposing $\psi_p^{01}(k)$ i.e. eqn. (V.42). This gives:

$$\psi_{po}^{01}(k) = \frac{\sqrt{2}}{(2\pi)^3}\frac{\delta(p-k)}{k^2}+\frac{M\sqrt{2}}{(2\pi)^3}\frac{T_o^{01}(k,p)}{p^2-k^2+i\eta} \tag{V.44}$$

The notation has been modified slightly in (V.44), in that the subscript o now stands for $\ell = 0$. The rest of the partial wave decomposition goes through as before (App. C),

and the $\ell = 0$ part of the potential is

$$W_{po}(q,k) = 2\pi \int_{-1}^{1} \tilde{\tilde{W}}\left[\frac{q^2+k^2-2p^2}{2M}; \vec{q}+\vec{k}\right] d(\cos \hat{q}\cdot\hat{k})$$

$$= -\frac{2f_o^2}{(2\pi)^3} + \frac{2f_o^2}{(2\pi)^3}\frac{1}{2kq}\left[\left(\frac{q^2+k^2-2p^2}{2M}\right)^2 - m^2\right] \times$$

$$\times \ell n \left| \frac{(q^2+k^2-2p^2)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2-2p^2)^2-4M^2(q+k)^2-4M^2m^2} \right| \qquad (V.45)$$

Eventually, the equation for $T_o^{01}(k,p)$ becomes

$$[p^2-k^2-G(p,k)]T_o^{01}(k,p) = -2\pi\,(p^2-k^2)\,W_{po}(p,k)$$

$$- 2\pi M(p^2-k^2)\int_0^\infty \frac{q^2dq}{p^2-q^2+i\eta} W_{po}(q,k)T_o^{01}(q,p)$$

which, upon defining $N(p,k)$ analogously to $M(p,k)$ i.e.

$$N(p,k) = \frac{p^2-k^2}{p^2-k^2-G(p,k)} \qquad (V.46)$$

becomes

$$T_o^{01}(k,p) = -2\pi N(p,k)W_{po}(p,k) - 2\pi MN(p,k) \times$$

$$\times \int_0^\infty \frac{q^2dq}{p^2-q^2+i\eta} W_{po}(q,k)T_o^{01}(q,p) \qquad (V.47)$$

For this particular channel, therefore, eqn. (V.47) is the s-wave equation, and can be compared to eqn. (IV.33). Although the integrals involved are more complicated, due to the structure of $W_{po}(q,k)$, the principle of solution is identical to that in the scalar model. There is no advantage to be had in discussing this in detail, except to mention that the approach is identical to that mentioned in App. D.

For completeness, the equation for the T-matrix in the other three channels can also be derived, and are included here:

$$T^{00}(\vec{k},\vec{p}) = -3\pi \frac{p^2-k^2}{p^2-k^2-G(p,k)} \left[\tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}-\vec{p}\right) + \tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}+\vec{p}\right) \right]$$

$$- 3M \frac{p^2-k^2}{p^2-k^2-G(p,k)} \int \frac{d^3q}{p^2-q^2+i\eta} \tilde{\tilde{W}}\left(\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right) T^{00}(\vec{q},\vec{p}) \qquad \text{(V.48)}$$

$$T^{10}(\vec{k},\vec{p}) = -3\pi \frac{p^2-k^2}{p^2-k^2-G(p,k)} \left[\tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}-\vec{p}\right) + \tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}+\vec{p}\right) \right]$$

$$- 3M \frac{p^2-k^2}{p^2-k^2-G(p,k)} \int \frac{d^3q}{p^2-q^2+i\eta} \tilde{\tilde{W}}_1\left(\frac{q^2+k^2-2p^2}{2M};\vec{q}+\vec{k}\right) T^{10}(\vec{q},\vec{p}) \qquad \text{(V.49)}$$

$$T^{11}(\vec{k},\vec{p}) = -3\pi \frac{p^2-k^2}{p^2-k^2-G(p,k)} \left[\tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}-\vec{p}\right) + \tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}+\vec{p}\right)\right]$$

$$- 3M \frac{p^2-k^2}{p^2-k^2-G(p,k)} \int \frac{d^3q}{p^2-q^2+i\eta} \tilde{\tilde{W}}_1\left(\frac{q^2+k^2-2p^2}{2M};q+k\right) T^{11}(\vec{q},\vec{p}) \qquad (V.50)$$

Note that $T^{00}(\vec{k},\vec{p}) = T^{00}(-\vec{k},\vec{p})$

$$T^{10}(\vec{k},\vec{p}) = T^{10}(-\vec{k},\vec{p})$$

$$T^{11}(\vec{k},\vec{p}) = -T^{11}(-\vec{k},\vec{p})$$

$$T^{01}(\vec{k},\vec{p}) = -T^{01}(-\vec{k},\vec{p})$$

The corresponding partial-wave decomposed equations read:

$$T_\ell^{00}(k,p) = -6\pi N(p,k)\, W_{p_\ell}(p,k) - 6\pi MN(p,k) \times$$

$$\times \int_0^\infty \frac{q^2 dq}{p^2-q^2+i\eta} W_{p_\ell}(q,k) T_\ell^{00}(q,p) \qquad (V.51)$$

and

$$T_\ell^{10}(k,p) = -6\pi N(p,k)\, W_{p_\ell}(p,k) - 6\pi MN(p,k) \times$$

$$\times \int_0^\infty \frac{q^2 dq}{p^2-q^2+i\eta} W_{p_\ell}^{(1)}(q,k) T_\ell^{00}(q,p) \qquad (V.52)$$

with an identical equation for $T_\ell^{11}(k,p)$.

V.4 Born Approximation

As in the scalar case, one can obtain the Born term in the S = 0, T = 1 channel. Once again, the self-energy is dropped, i.e. N(p,k) is made equal to I. Then, as before (eqn. (IV.35)), the Born term is

$$T_{PS}^{(1)}(p) = -\pi \lim_{k\to p} \left[\tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}-\vec{p}\right) + \tilde{\tilde{W}}\left(\frac{k^2-p^2}{2M};\vec{k}+\vec{p}\right) \right]$$

$$= -\pi \left[\tilde{\tilde{W}}(o,\vec{k}-\vec{p}) + \tilde{\tilde{W}}(o,\vec{k}+\vec{p})\right]$$

$$= -\frac{\pi f_o^2}{(2\pi)^4}\left[\frac{(\vec{k}-\vec{p})^2}{-(\vec{k}-\vec{p})^2-m^2} + \frac{(\vec{k}+\vec{p})^2}{-(\vec{k}+\vec{p})^2-m^2}\right]$$

$$= \frac{\pi f_o^2}{(2\pi)^4}\left[1 - \frac{m^2}{m^2+|\vec{k}-\vec{p}|^2} + 1 - \frac{m^2}{m^2+|\vec{k}+\vec{p}|^2}\right]$$

$$= \frac{2\pi f_o^2}{(2\pi)^4} - \frac{\pi f_o^2 m^2}{(2\pi)^4}\left[\frac{1}{m^2+|\vec{k}-\vec{p}|^2} + \frac{1}{m^2+|\vec{k}+\vec{p}|^2}\right] \qquad (V.53)$$

Eqn. (V.53) indicates that the Born term contains, in addition to an attractive Yukawa-type term, a constant term. In configuration space, this constant arises from a delta-function at the origin which, for local potentials, has no effect on the wavefunction. However, in our non-local case, the δ-function appears to get spread out, and since the sign of the term is positive, a repulsive core seems to be present. So the Born term appears to arise from a repulsive

core and an attractive Yukawa potential. This feature is quite different from the earlier model, which was purely attractive. One would expect the repulsion to "pull down" the phase shifts accordingly.

V.5 The Deuteron

The deuteron is predominantly in an s-state; however, the existence of a non-zero quadrupole moment, and the value of the magnetic moment indicate the presence of a d-wave admixture. This actually requires the presence of a tensor term in any potential which is supposed to describe the deuteron.

Assuming that the deuteron wavefunction is a mixture of s- and d-waves, one can derive coupled equations for the two components. This leads to the Rarita-Schwinger equations[30] first obtained in 1941. A similar approach can be taken in our case. The main difference will, of course, be that the potentials (central and tensor) will be non-local, and in principle, determined, whereas the potentials are usually inputs in the conventional treatment[31].

The deuteron wavefunction satisfies eqn. (V.31) which, upon subtracting out the self-energy, becomes

$$(\nabla_r^2 - ME) f_b^{10}(\vec{r}) = - 6\pi M \int d^3r' \; W_o(\vec{r}-\vec{r}') f_b^{10}(\vec{r}')$$

$$+ \frac{3}{2}M\int d^3r'd^3q W_{Eq}(\vec{r}-\vec{r}')\,[f_q^{01}(\vec{r})\,f_q^{01*}(\vec{r}')+f_q^{11}(\vec{r})\,f_q^{11*}(\vec{r}')]\,f_b^{10}(\vec{r}')$$

$$- \frac{3}{2}M\int d^3r'd^3q W_{Eq}^{(1)}(\vec{r}-\vec{r}')\,[f_q^{01}(\vec{r})\,f_q^{01*}(\vec{r}')-f_q^{11}(\vec{r})\,f_q^{11*}(\vec{r}')]\,f_b^{10}(\vec{r}')$$

(V.54)

The term involving $W_{Eq}^{(1)}$ will introduce a tensor term into the potential. (V.54) is first rewritten as

$$(\nabla_r^2 - ME)\,f_b^{10}(\vec{r}) = -6\pi M\int d^3r'\; W_o(\vec{r}-\vec{r}')\,f_b^{10}(\vec{r}')$$

$$+ \frac{3}{2}M\int d^3r'd^3q\,[W_{Eq}(\vec{r}-\vec{r}') - W_{Eq}^{(1)}(\vec{r}-\vec{r}')]\,f_q^{01}(\vec{r})\,f_q^{01*}(\vec{r}')\,f_b^{10}(\vec{r}')$$

$$+ \frac{3}{2}M\int d^3r'd^3q\,[W_{Eq}(\vec{r}-\vec{r}') + W_{Eq}^{(1)}(\vec{r}-\vec{r}')]\,f_q^{11}(\vec{r})\,f_q^{11}(\vec{r}')\,f_b^{10}(\vec{r}')$$

(V.55)

But we have that

$$W_{Eq}(\vec{x}) \pm W_{Eq}^{(1)}(\vec{x}) = f_o^2\int d^3k\; e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\;\frac{k^2 \pm (\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})}{\left(\frac{q^2}{M}+\frac{k^2}{4M}+E_b\right)^2 - k^2 - m^2}$$

(V.56)

from the definitions of W and $W^{(1)}$. In (V.56), the tensor operator is introduced via the relationship

$$k^2 S_{12} = 3(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k}) - k^2(\vec{\sigma}_1\cdot\vec{\sigma}_2)$$

So the $(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})$ term is replaced by a tensor term, and the $(\vec{\sigma}_1\cdot\vec{\sigma}_2)$ term adds to the central part of (V.56). Eventually, therefore, (V.55) becomes:

$$(\nabla_r^2 - ME) f_b^{10}(\vec{r}) = -6\pi M \int d^3r'\, W_o(\vec{r}-\vec{r}') f_b^{10}(\vec{r}')$$

$$+ \tfrac{3}{2}M \int d^3r'\, d^3q\, W_{Eq}(\vec{r}-\vec{r}')\{[1 - \tfrac{1}{3}(\vec{\sigma}_1\cdot\vec{\sigma}_2)] f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}')$$

$$+ [1 + \tfrac{1}{3}(\vec{\sigma}_1\cdot\vec{\sigma}_2)] f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')\} f_b^{10}(\vec{r}')$$

$$+ \tfrac{3}{2}M \int d^3r'\, d^3q\, W_{Eq}(\vec{r}-\vec{r}')\{ -\tfrac{1}{3} f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}')$$

$$+ \tfrac{1}{3} f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')\} S_{12} f_b^{10}(\vec{r}') \ .$$

which may be written symbolically as

$$(\nabla_r^2 - ME) f_b^{10}(\vec{r}) = \int d^3r' V_C(\vec{r},\vec{r}') f^{10}(\vec{r}') + \int d^3r' V_T(\vec{r},\vec{r}') S_{12} f_b^{10}(\vec{r}') \qquad (V.57)$$

Eqn. (V.57) is a Schrödinger equation with non-local central and tensor forces $V_C(\vec{r},\vec{r}')$ and $V_T(\vec{r},\vec{r}')$ respectively, which are defined as follows:

$$V_C(\vec{r},\vec{r}') = -6\pi M\, W_o(\vec{r}-\vec{r}')$$

$$+ \tfrac{3}{2}M \int d^3q\, W_{Eq}(\vec{r}-\vec{r}')\{[1 - \tfrac{1}{3}(\vec{\sigma}_1\cdot\vec{\sigma}_2)] f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}')$$

$$+ [1 + \tfrac{1}{3}(\vec{\sigma}_1 \cdot \vec{\sigma}_2)] f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')\} \qquad (V.58)$$

and

$$V_T(\vec{r},\vec{r}') = -\tfrac{1}{2}M \int d^3q W_{Eq}(\vec{r}-\vec{r}') [f_q^{01}(\vec{r}) f_q^{01*}(\vec{r}') - f_q^{11}(\vec{r}) f_q^{11*}(\vec{r}')] \qquad (V.59)$$

Eqn. (V.59) is now decomposed into partial waves. Since the deuteron is in a spin triplet state, $(\vec{\sigma}_1 \cdot \vec{\sigma}_2) = 1$. Following the method and notation of DeBenedetti[32], (where the analysis for purely local potentials is done), we can write the deuteron wavefunction as a combination of s- and d-wave components, i.e.

$$\mathcal{Y}^1_{101}(\hat{r}) \frac{f_S(r)}{r} + \mathcal{Y}^1_{121}(\hat{r}) \frac{f_D(r)}{r} \qquad (V.60)$$

where $f_S(r)$ and $f_D(r)$ are the s- and d-wave parts, and

$$\mathcal{Y}^1_{101}(\hat{r}) = Y_{00}(\hat{r}) \chi^{11} \qquad (V.61)$$

$$\mathcal{Y}^1_{121}(\hat{r}) = \sqrt{\tfrac{6}{10}}\, Y_{22}(\hat{r}) \chi^{1-1} - \sqrt{\tfrac{3}{10}}\, Y_{21}(\hat{r}) \chi^{10} + \sqrt{\tfrac{1}{10}}\, Y_{20}(\hat{r}) \chi^{11} \qquad (V.62)$$

The $\chi^{\alpha\beta}$'s are the spin wavefunctions, and the $Y_{\ell m}$'s the usual spherical harmonics. If we assume that the deuteron

is in fact described by (V.61), we can decompose eqn. (V.57). In order to do this, we need the following:

$$S_{12}\mathcal{Y}^1_{101} = \sqrt{8}\,\mathcal{Y}^1_{121} \quad ; \quad S_{12}\mathcal{Y}^1_{121} = \sqrt{8}\mathcal{Y}^1_{101} - 2\mathcal{Y}^1_{121}$$

$$(\vec{\sigma}_1\cdot\vec{\sigma}_2)\mathcal{Y}^1_{1n1} = \mathcal{Y}^1_{1n1} \quad ; \quad n = 0,\ 2.$$

Then we are led to the following coupled equations:

$$\frac{d^2 f_S(r)}{dr^2} - ME f_S(r) = \int_0^\infty r'^2 dr' V_C(r,r') f_S(r') + \sqrt{8}\int_0^\infty r'^2 dr' V_T(r,r') f_D(r') \qquad (V.63)$$

$$\frac{d^2 f_D(r)}{dr^2} - \frac{6 f_D(r)}{r^2} - ME f_D(r) = \int_0^\infty r'dr'[V_C(r,r') - 2V_T(r,r')] f_D(r') + \sqrt{8}\int_0^\infty r'^2 dr' V_T(r,r') f_S(r') \qquad (V.64)$$

Eqns. (V.63) and (V.64) are identical to the Rarita-Schwinger equations, except that the potentials are non-local (and inherently so), in this case. Both $V_C(r,r')$ and $V_T(r,r')$ may be written down, by just partial wave decomposing the original $V_C(\vec{r},\vec{r}')$ and $V_T(\vec{r},\vec{r}')$, i.e. eqns. (V.58) and (V.59). So if, as in the scattering channels, one uses free solutions for the intermediate states, one has,

in principle, a non-local potential describing the deuteron. Once again, however, the integrals cannot be evaluated in closed form. The point to note is that instead of stipulating that V_C and V_T (in the local case) are, for example, square wells, or other simple potentials, we have, in fact, a potential derived from the theory. Examination of (V.58), for example, will show that, even in the local limit, there is a great deal of structure in the potentials. Although square well potentials are used for simplicity, it would seem that one actually needs potentials of a far more complicated nature to describe the deuteron.

FIGURE II

PSEUDOSCALAR MODEL: SOME RESULTS

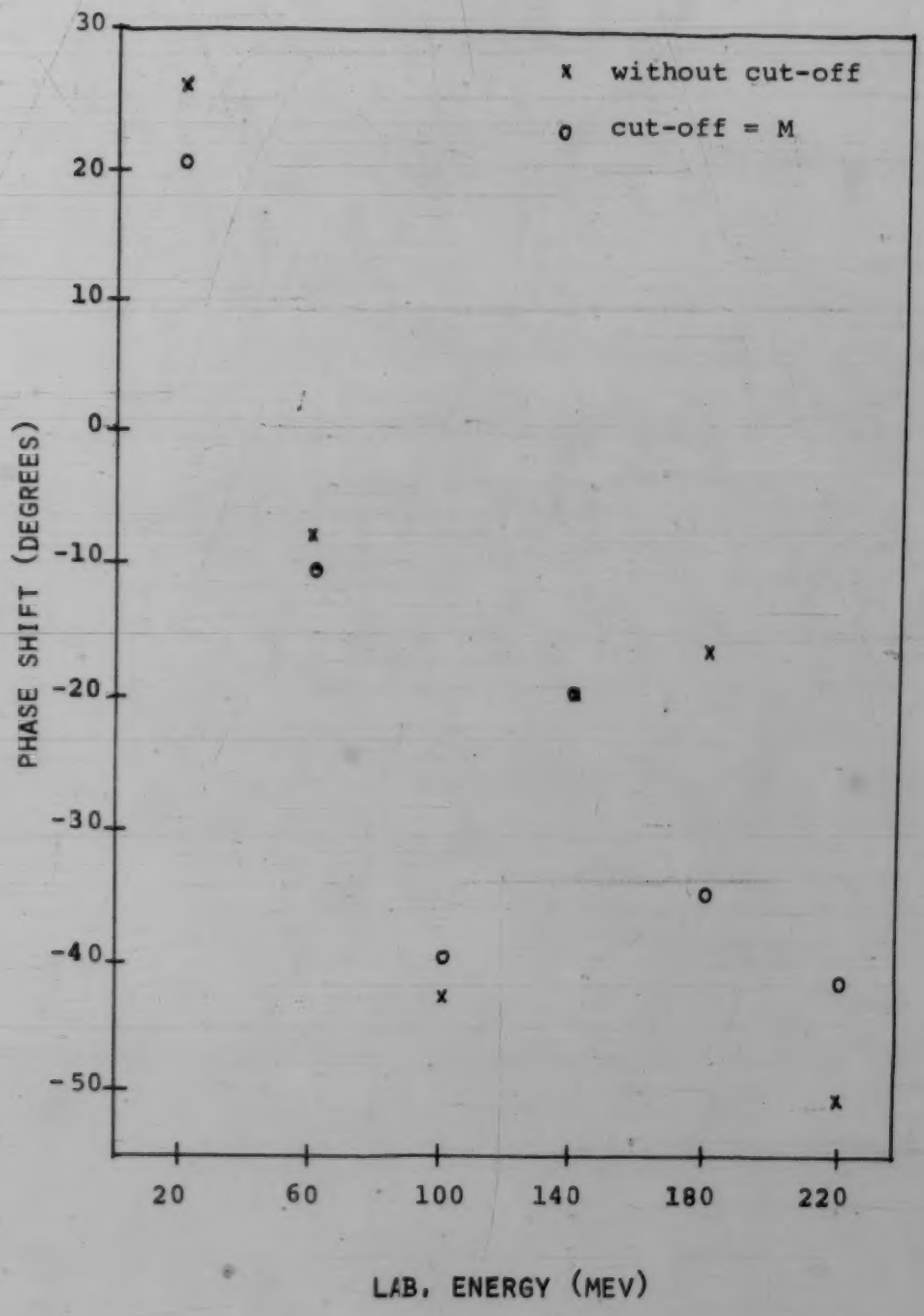


FIGURE III

SCALAR MODEL: PHASE SHIFTS ($g^2 = 2.98$)

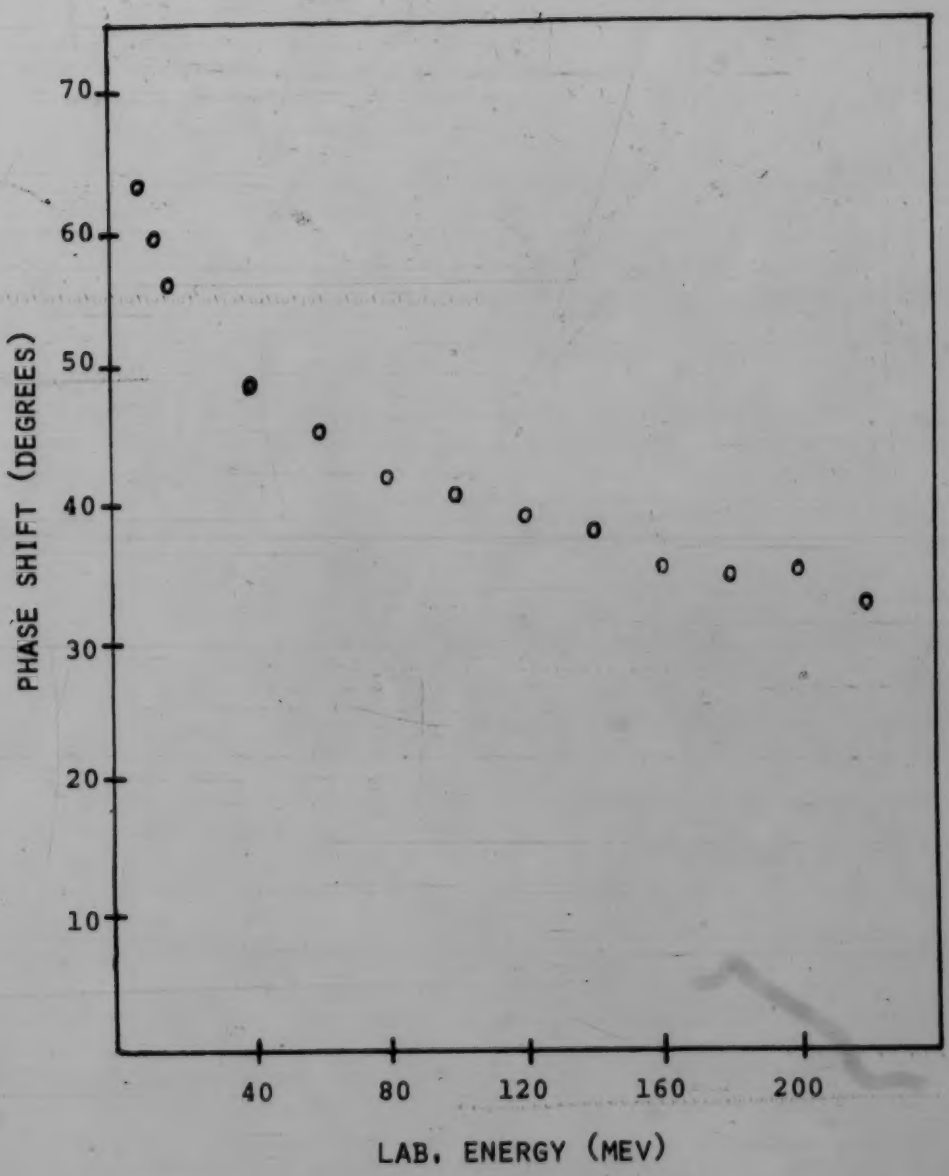
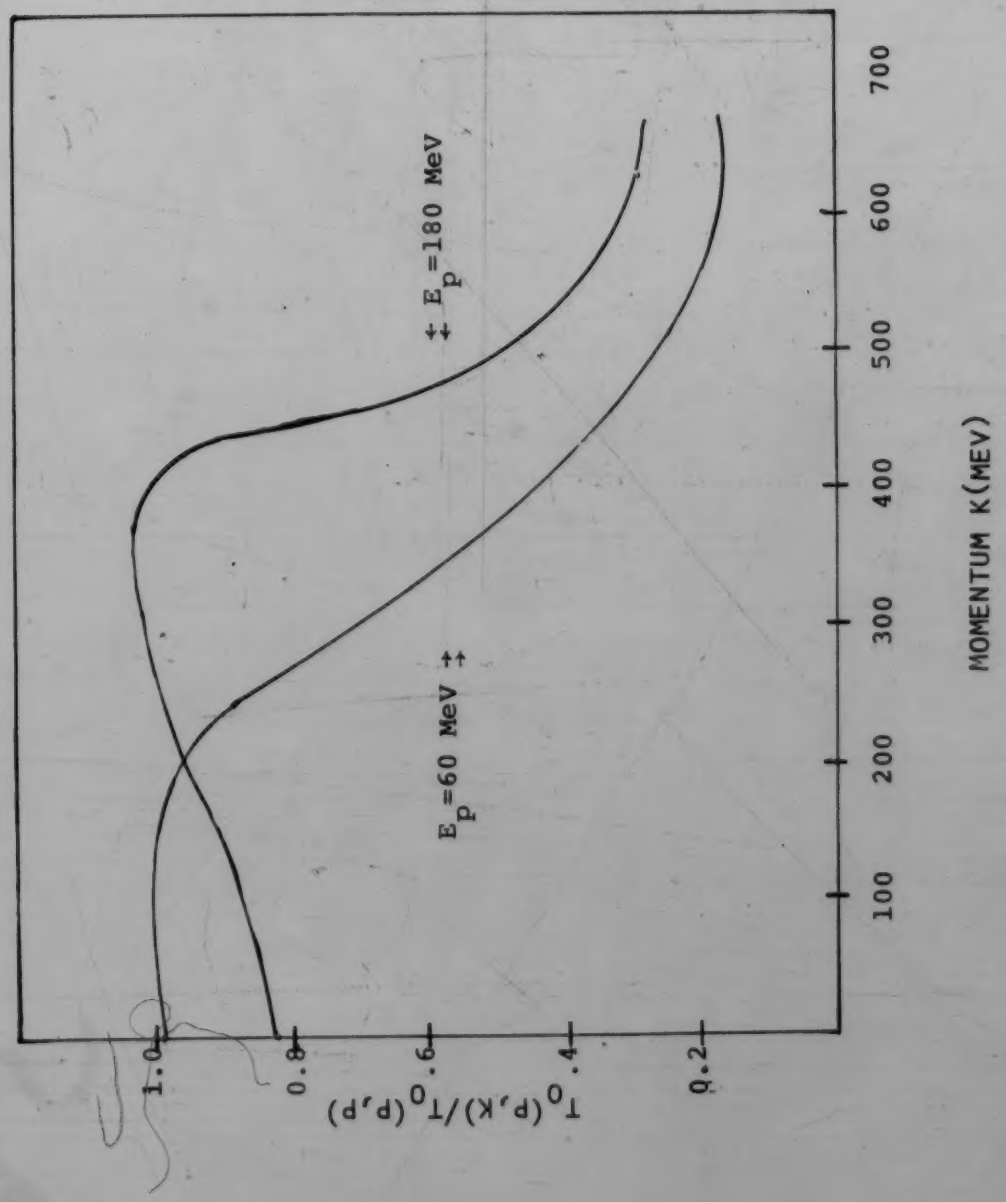


FIGURE IV

HALF OFF SHELL T MATRIX ELEMENTS



VI. DISCUSSION OF RESULTS AND CONCLUSIONS

The technique that has been employed here had its roots in the solution of the anharmonic oscillator problem[34]. Further investigations, particularly into the non-relativistic few-body problem[22] indicated that this was perhaps applicable to the interaction between two nucleons. Field theory has, of course, been used to tackle this problem before, and the vast quantity of literature attests to this fact. Most of the previous work, however, has been perturbation-theoretically inclined.

The applicability of this particular scheme to the current problem depends upon two facts. Firstly, the equations of motion (or the Lagrangian, for that matter) are of a very particular type; the "source" term for one of the fields (the meson field in this case) is entirely independent of the field itself; i.e. there are no direct meson-meson interactions. This enables the fields to decouple (at least formally). Secondly, the models are non-relativistic, and our interest is confined to (at most) two-particles. This few-body aspect enables the different particle-number states to partially decouple; i.e. for a solution of the two-body system, only one- and two-particle states need to be considered.

We examined first, mainly for purposes of developing the methods required, a model in which the non-relativistic nucleons interact via the exchange of spinless, chargeless

mesons. This model, the scalar model, is far more tractable than a more physical one, not only because of the absence of spin and isospin, but also due to the simpler structure of the kernels that appear in the final Lippmann-Schwinger equations.

By considering single particles in this theory, we saw that "free" wavefunctions, which form a complete set, do describe them. This implies, however, that the usual non-relativistic energy-momentum relationship

$$E_p = \frac{p^2}{2M}$$

can no longer be true. There is an additional term, an explicit form of which can only be obtained, in principle, on solving a non-linear integral equation for the energy. This extra term is interpreted as a self-energy; such an interpretation seems quite natural, at least for single particles, as there can be no other interactions present. Due to the complexity of the integral equation for E_p, we have not solved it explicitly; instead, we have identified the integral term i.e. the difference between E_p and the "free" energy $\frac{p^2}{2M}$ as the self-energy.

In the two-particle case, equations are obtained with "potentials" which are both non-local and energy-dependent. There are only two channels possible in the scalar model, viz. scattering states and bound states. The scattering states were examined in both configuration

space and momentum space. In configuration space, an integral representation of the non-local potential was obtained; it is not possible to write down a closed, exact form of the potential in configuration space. From the momentum space equation, an equation for the half-off-shell T-matrix, $T(\vec{k},\vec{p},p^2)$, was derived. This is, again, an integral equation, and from the inhomogeneous term, the Born approximation was obtained. This term is identical, to within factors involving the coupling constant, to that obtained from a local Yukawa potential; in this approximation, at least, the potential is attractive, due to our "strong-arming" the coupling constant. On shell, it is in fact a Yukawa potential.

The s-wave T-matrix equation was examined further. Half-off-shell matrix elements were obtained; a lengthy algebraic and numerical procedure was required to do this. From the corresponding on-shell values, phase shifts were calculated. These phase shifts are plotted on Figure I, for three different combinations of parameters. The phase shift starts at 180°, and so indicates the presence of a bound state (the deuteron). After an initial rapid drop, the phase shift seems to level off at rather a high value (~ 60°). This would seem to indicate that there is not quite enough repulsion in this model. If our treatment of this model is to fit experimental data reasonably well, and if the phase shifts are to drop quicker (and, in fact,

take on negative values), repulsion would have to be incorporated in the form of a core.

The results we obtain seem to indicate a good solution of the Lippmann-Schwinger equation; there is numerical consistency, and very little sensitivity to the value (or even the presence) of a cut-off. Figure I also shows that the inclusion of self-energy does not alter the results by much; this supports the contention that self-energy can, for all practical purposes, be neglected.

The numerical results also satisfy unitarity very well. Any deviations from exact unitarity are far too small to be anything but numerical (roundoff) in origin. In any case, we would expect unitarity to hold, as the potentials are all real. Also, the energies we are concerned with are low enough that only elastic processes are involved; therefore, the phase shifts should all be real. Numerically, the calculation yields values of the order of 10^{-13} - 10^{-14} radians for the imaginary part of the phase shift; this can be compared to values of the order of 1 or 2 radians for the real part.

As for the bound state in the scalar model, the treatment is slightly different. Using a trial wavefunction (a single exponential in this case), we look for a minimum in binding energy. Any possible refinement on the accuracy of the results by the use of a more complicated wavefunction (e.g. the Hulthen wavefunction) was far outweighed by the

added complexity of the calculation on doing so. The calculations exhibit a large degree of sensitivity to the value of g^2, the coupling constant. By constraining the value of the parameter of the wavefunction (or equivalently, the range of the exponential) to be close to $\sqrt{ME}$, a value of $g^2 = 4.6$ was obtained. This was the value of g^2 used in the phase shift calculations, and also in the discussion of the Born term.

One further point should be made regarding the scalar theory. The self-energy term was examined in both the zero and infinite momentum limits. In the former case, it showed that a renormalization of a few percent was all that was needed to correct for self-energy; on the other hand, the infinite renormalization is identically zero. These calculations have not been included; instead, the corresponding pseudoscalar calculations appear in Appendix F.

The second model considered is the pseudoscalar one, where both spin and isospin are included. Apart from the additional matrix algebra required, the one-particle system is treated identically to that in the scalar model. Once again, a self-energy term can be derived.

The two-nucleon system is quite a bit more complicated. There are four possible channels in which scattering takes place, and the single (deuteron) bound state. Equations here obtained for all five wavefunctions; also,

the T-matrix equations were obtained for the scattering channels. The potentials are more involved now, and in some channels (particularly the deuteron one), tensor forces appear, as they should.

One of the simpler scattering channels is the S = 0, T = 1 channel. There is no tensor force present, and the method employed for the scalar model should carry through. But if one compares the equation for the T-matrix in this channel with the one in scalar theory, it can be seen that the kernel differs from the scalar one by a polynomial factor; also there are two (separately) divergent terms in the (0,1) channel which, together, will cancel out.

Numerically, the results we obtained in this case are not as good as in the earlier case. Whereas, for the scalar case, numerical consistency was obtained using 30 x 30 and 40 x 40 matrices, we had to use 60 x 60 matrices before a solution begins to appear (and sensitivity to the cutoff begins to vanish). This could arise from a number of things, one being increased structure in the T-matrix. In solving the equation, we have used linear interpolation of the T's; perhaps this is not enough to account for the structure. Taking a larger grid size circumvents this difficulty.

Using the physical coupling constant ($f_o^2/(2\pi)^3 = 0.189$) we did obtain a solution (see Fig. II); however, it still seems to contain too much structure. Also, it drops too

rapidly through zero. As opposed to scalar theory, therefore, there is obviously too much repulsion here. The calculations were repeated with various values of coupling constant, but this does not alter the degree of repulsion present.

These difficulties seem to be tied in with a more general problem. Theoretically, it is exceedingly difficult to describe the s-wave interaction well. At short distances, exceedingly complicated particle exchanges come into play; this does not occur for higher partial waves. In fact, the problem can be seen quite clearly when one considers that the origin of the repulsive core came from the fact that the ^1S_o phase shift goes negative - a fact that cannot be explained otherwise. It is apparent, therefore, that the difficulty of handling the ^1S_o state adequately has been around for a while. In particular, the amount of repulsion needed is one aspect that has yet to be treated completely, theoretically.[35]

The two models therefore seem to differ basically in the amount of repulsion present. The Born terms on-shell, for example, differ basically by a constant. This constant, which is analogous, in the local case, to a δ-function potential at the origin, in this non-local case creates a sort of core, thereby causing the repulsion. With this in mind, a calculation was performed in the scalar theory, to see whether the ^1S_o data can be reproduced. (Fig. III) . The value of g^2 was 2.98, and the phase shift

drops from about 60^0. However, once again, δ_o stays non-negative up to at least 280 MeV. In addition to the phase shifts, the off-shell behaviour of T was examined. In Fig. IV, the matrix elements $T_o(p,k)$ (normalised to 1) are plotted for two values of E_p, viz. 60 MeV and 180 MeV. Typically, these curves should go through zero; but because the phase shifts stay positive, so do these ratios.

In the pseudoscalar model, this difficulty seems to be present only in the s-wave. This is primarily due to the constant term in the Born approximation mentioned in the previous paragraph. However, upon partial wave decomposing, this constant only contributes to the s-wave. Therefore, for higher partial waves, there will probably be improvement. A preliminary calculation, using only the Born term, seems to confirm this.

At this stage, it is safe to say that we have a numerical technique that enables us to obtain a half-off-shell T-matrix, and corresponding phase shifts, from a theory such as this. However, only $\ell=0$ has been considered, and though generalisation to $\ell=2$, etc. is straightforward, it is far from trivial. The structure of the kernels become far more complicated. Although the numerics appear to be acceptable, the physics in the model does not give very good agreement with experiments, at least for s-waves. But, as mentioned before, this probably stems from the nature of the s-wave interaction itself. Theory appears

to keep having a very difficult time with theory.

Inherently, this technique is of value- qualitative features of the N-N interaction seem to arise naturally: a repulsive core; the tensor force in the deuteron; different potentials in the various channels, etc. Quantitatively, it enables one, in principle, to evaluate half-off-shell T-matrices. This in itself probably justifies further work, as lately, such calculations have been increasing considerably in number.

As for the non-local potentials themselves (e.g. in the deuteron), both non-locality and energy-dependence appear naturally. However, the structure of these potentials lead us to believe that no local approximation is going to take account enough features of the original potential. Such approximations may be useful in very weakly or very strongly non-local potentials; here, the potentials are far too complex to be so categorised.

For scattering, this technique should be applied to higher partial waves. If, as we speculate, the problems we have encountered arise from the nature of thes-wave itself, a calculation for $\ell=2$ would certainly produce better results. Also, it would be worthwhile to develop a method for solving all the four coupled equations. In principle, a (much) larger grid would permit this to be done. As for the bound state, the non-local potentials V_C and V_T should

be examined more closely. The possibility of a numerical solution (in an iterative manner, perhaps) of the deuteron should be looked at. Whether one can obtain a reasonably good approximation to the (now) complicated potentials is a question well worth pursuing. There seems to be, at this stage, anyway, more work needed to see whether this theory will stand or fall.

APPENDIX A: EVALUATION OF THE POTENTIAL

$$I = \int d^3q\, V_{pq}(\vec{x}) e^{-i\vec{q}\cdot\vec{y}}$$

$$= (2\pi)^4 \int d^3q\, d^3k\, e^{-i\vec{q}\cdot\vec{y}}\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\, \tilde{V}\left[\frac{4q^2-4p^2+k^2}{4M}\,;\vec{k}\right]$$

$$= g^2 \int d^3k\, d^3q\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}}\, e^{-i\vec{q}\cdot\vec{y}}\, \frac{1}{\left(\frac{4q^2-4p^2+k^2}{4M}\right)^2 - k^2 - m^2}$$

$$= 16M^2g^2 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \int d^3q\, \frac{e^{-i\vec{q}\cdot\vec{y}}}{(4q^2-4p^2+k^2)^2 - 16M^2k^2 - 16M^2m^2} \,. \qquad \text{(A1)}$$

Consider

$$D = (4q^2-4p^2+k^2)^2 - 16M^2k^2 - 16M^2m^2$$

$$= 0 \quad \text{when} \quad q_\pm^2 = p^2 - \frac{k^2}{4} \pm M\sqrt{k^2+m^2} \,. \qquad \text{(A2)}$$

The various possible cases have to be considered separately, to determine the possible signs of $q_\pm^2$.

Case (i) $k < 2p$:

In this case, $p^2 > \frac{k^2}{4}$

$\therefore\ q_+^2 = p^2 - \frac{k^2}{4} + M\sqrt{k^2+m^2} > 0$ always

But q_-^2 could change sign;

$$q_-^2 = p^2 - \frac{k^2}{4} - M\sqrt{k^2+m^2} \gtrless 0 \text{ when}$$

$$p^2 - \frac{k^2}{4} \gtrless M\sqrt{k^2+m^2} \quad .$$

Now,

$$(p^2 - \frac{k^2}{4})^2 - M^2(k^2+m^2) = 0$$

when

$$k^2 = 8(M^2 + \tfrac{1}{2} p^2) \pm 8M^2 \sqrt{1 + \frac{p^2}{M^2} + \frac{m^2}{4M^2}} \quad .$$

The energies one is interested in are below the one-pion threshold, i.e. $p^2 \ll mM$; in this range

$$8(M^2 + \tfrac{1}{2} p^2) < 8M^2 \sqrt{1 + \frac{p^2}{M^2} + \frac{m^2}{4M^2}} \quad ,$$

so the only positive solution for k^2 is

$$k^2 = 8M^2 \left(1 + \frac{p^2}{2M^2}\right) + 8M^2 \sqrt{1 + \frac{p^2}{M^2} + \frac{m^2}{4M^2}} = \mu_p^{\,2}$$

when $\quad k^2 \lesssim \mu_p^{\,2}, \quad p^2 - \frac{k^2}{4} \lesssim M\sqrt{k^2+m^2}$

$$\text{i.e.} \quad q_-^2 \lesssim 0 \quad .$$

But since case (i) is restricted to $k < 2p$, and since $\mu_p > 2p$, one is left with the case $k^2 < \mu_p^2$; therefore, $q_-^2 < 0$

$$\therefore Q_-^2 = -q_-^2 = -p^2 + \frac{k^2}{4} + M\sqrt{k^2+m^2} > 0$$

In case (i),

$$D = (q^2-q_+^2)(q^2+Q_-^2) \quad . \tag{A3}$$

<u>Case (ii)</u> $k > 2p$ or $\frac{k^2}{4} > p^2$

From (A2), it is obvious that $q_-^2 < 0$ always

$$q_+^2 \gtrless 0 \quad \text{when} \quad p^2 - \frac{k^2}{4} \gtrless -M\sqrt{k^2+m^2}$$

Since $p^2 - \frac{k^2}{4} < 0$, this implies that $\frac{k^2}{4} - p^2 \lesssim M\sqrt{k^2+m^2}$

For $k^2 \gtrless \mu_p^2$, $\quad (\frac{k^2}{4} - p^2)^2 \gtrless M^2(k^2+m^2)$

$$\text{i.e. } \frac{k^2}{4} - p^2 \gtrless M\sqrt{k^2+m^2}$$

$$\text{i.e. } q_+^2 \lesssim 0$$

$$\therefore \text{for } k^2 < \mu_p^2, \quad q_+^2 = p^2 - \frac{k^2}{4} + M\sqrt{k^2+m^2} > 0$$

$$k^2 > \mu_p^2, \quad Q_+^2 = -q_+^2 = -p^2 + \frac{k^2}{4} - M\sqrt{k^2+m^2} > 0$$

$\therefore$ Case (ii):

$$D = (q^2+Q_-^2)(q^2-q_+^2)\theta(\mu_p-k) + (q^2+Q_-^2)(q^2+Q_+^2)\theta(k-\mu_p)$$

So:

$$D = (q^2-q_+^2)(q^2+Q_-^2) \qquad k < \mu_p$$

$$= (q^2+Q_+^2)(q^2+Q_-^2) \qquad k > \mu_p \ .$$

Therefore (A1) can be rewritten:

$$I = 16M^2g^2 \int d^3k\, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \int d^3q \left(\frac{e^{-i\vec{q}\cdot\vec{y}}}{(q^2-q_+^2)(q^2+Q_-^2)}\,\theta(\mu_p-k) + \frac{e^{-i\vec{q}\cdot\vec{y}}}{(q^2+Q_+^2)(q^2+Q_-^2)}\,\theta(k-\mu_p)\right) . \tag{A4}$$

Define

$$I_< = \int d^3q\, \frac{e^{-i\vec{q}\cdot\vec{y}}}{(q^2-q_+^2)(q^2+Q_-^2)} = \frac{4\pi}{y}\int_0^\infty qdq\, \frac{\sin(qy)}{(q^2-q_+^2)(q^2+Q_-^2)}$$

$$= \frac{2\pi^2}{q_+^2+Q_-^2}\left(\frac{\cos(q_+y)-e^{-Q_-y}}{y}\right) = \frac{\pi^2}{M\sqrt{k^2+m^2}}\left(\frac{\cos(q_+y)-e^{-Q_-y}}{y}\right) .$$

$$I_> = \int d^3q \, \frac{e^{-i\vec{q}\cdot\vec{y}}}{(q^2+Q_-^2)(q^2+Q_-^2)} = \frac{4\pi}{y} \int_0^\infty qdq \, \frac{\sin(qy)}{(q^2+Q_+^2)(q^2+Q_-^2)}$$

$$= \frac{2\pi^2}{Q_+^2-Q_-^2} \left(\frac{e^{-Q_-y} - e^{-Q_+y}}{y} \right) = \frac{\pi^2}{M\sqrt{k^2+m^2}} \left(\frac{e^{-Q_+y} - e^{-Q_-y}}{y} \right) .$$

$$\therefore I = \frac{16\pi^2 Mg^2}{y} \int d^3k \, e^{-i\frac{\vec{k}}{2}\cdot\vec{x}} \, \frac{1}{\sqrt{k^2+m^2}} \, \mathrm{Re}\left(e^{-Q_+y} - e^{-Q_-y} \right)$$

$$= \frac{64\pi^3 Mg^2}{xy} \int_0^\infty \frac{kdk \, \sin(\frac{k}{2}x)}{\sqrt{k^2+m^2}} \, \mathrm{Re}\left(e^{-Q_+y} - e^{-Q_-y} \right)$$

$$= \frac{8(2\pi)^3 Mg^2}{xy} \, \mathrm{Re} \int_0^\infty dk \, \frac{k \, \sin(\frac{k}{2}x)}{\sqrt{k^2+m^2}} \left(e^{-Q_+y} - e^{-Q_-y} \right) .$$

(A5)

APPENDIX B: TRANSFORMATION INTO MOMENTUM SPACE

Rewrite eqns. (IV.14) and (IV.24):

$$(\nabla_r^2 + p^2) f_p(\vec{r}) = -2\pi M \int d^3r'\, V_o(\vec{r}-\vec{r}') f_p(\vec{r}') + M \int d^3r' \left[\int d^3q\, V_{pq}(\vec{r}-\vec{r}') f_q^{(o)}(\vec{r}) f_q^{(o)*}(\vec{r}') \right] f_p(\vec{r}') \tag{B1}$$

and

$$f_p(\vec{r}) = \int d^3k\, e^{-i\vec{k}\cdot\vec{r}}\, \psi_p(\vec{k}). \tag{B2}$$

Substituting (B2) into (B1) yields

$$\int e^{-i\vec{k}\cdot\vec{r}} (-k^2+p^2) \psi_p(\vec{k}) d^3k = -2\pi M \int V_o(\vec{r}-\vec{r}')\, e^{-i\vec{k}\cdot\vec{r}'} \psi_p(\vec{k})\, d^3k\, d^3r' + M \int d^3r' \left[\int d^3q\, V_{pq}(\vec{r}-\vec{r}') \psi_q^{(o)}(\vec{k}_1) \psi_q^{(o)*}(\vec{k}_2) \right] \times \times \psi_p(\vec{k}_3)\, e^{-i(\vec{k}_1\cdot\vec{r}-\vec{k}_2\cdot\vec{r}'+\vec{k}_3\cdot\vec{r}')}\, d^3k_1\, d^3k_2\, d^3k_3$$

i.e.

$$(p^2-k^2)\psi_p(\vec{k}) = -2\pi M \int d^3y\, V_o(\vec{y})\, e^{-i\vec{k}\cdot\vec{y}}\, \psi_p(\vec{k})$$

$$+ M \int \tilde{V}_{pq}(\vec{k}_2-\vec{k}_3)\psi_q^{(o)}(\vec{k}_1)\psi_q^{(o)*}(\vec{k}_2) \times$$

$$\times \psi_p(\vec{k}_3)\delta(\vec{k}-\vec{k}_1+\vec{k}_2-\vec{k}_3)\, d^3k_1\, d^3k_2\, d^3k_3\, d^3q \tag{B3}$$

where

$$\tilde{V}_{pq}(\vec{k}_2-\vec{k}_3) = \int V_{pq}(\vec{y}) e^{-i(\vec{k}_2-\vec{k}_3)\cdot\vec{y}}\, d^3y$$

$$= 8(2\pi)^7 \tilde{\tilde{V}}\left[\frac{(\vec{k}_2-\vec{k}_3)^2+q^2-p^2}{M};\ -2(\vec{k}_2-\vec{k}_3)\right] \tag{B4}$$

Defining

$$V_{-k} = \int d^3y\, e^{i\vec{k}\cdot\vec{y}}\, V_o(\vec{y}) = \int d^3q\, \tilde{\tilde{V}}[E_q-E_k;\vec{q}-\vec{k}] \tag{B5}$$

eqn. (B1) becomes

$$(p^2-k^2)\psi_p(\vec{k}) = -2\pi M\, V_{-k}\psi_p(\vec{k})$$

$$+ \int \tilde{V}_{pk_1}(\vec{k}_2-\vec{k}_3)\psi_{k_1}^{(o)}(\vec{k}+\vec{k}_2-\vec{k}_3)\psi_{k_1}^{(o)*}(\vec{k}_2)\psi_p(\vec{k}_3)d^3k_1d^3k_2d^3k_3. \tag{B6}$$

The free configuration space wavefunction is [eqns. (IV.16) and (IV.17)]

$$f_p^{(o)}(\vec{r}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[e^{i\vec{p}\cdot\vec{r}} + e^{-i\vec{p}\cdot\vec{r}}\right]. \qquad \text{(B7)}$$

From (B2) and (B7), one gets that

$$\psi_p^{(o)}(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[\delta(\vec{p}+\vec{k}) + \delta(\vec{p}-\vec{k})\right]. \qquad \text{(B8)}$$

Consider the rhs of (B6); when (B8) is inserted, one is led to, apart from the self-energy:

$$\frac{M}{2(2\pi)^6}\int \tilde{v}_{pk_1}(\vec{k}_2-\vec{k}_3)\left[\delta(\vec{k}+\vec{k}_1+\vec{k}_2-\vec{k}_3)+\delta(\vec{k}_1-\vec{k}-\vec{k}_2+\vec{k}_3)\right]\times$$

$$\times\left[\delta(\vec{k}_1+\vec{k}_2)+\delta(\vec{k}_1-\vec{k}_2)\right]\psi_p(\vec{k}_3)\; d^3k_1\; d^3k_2\; d^3k_3$$

$$= \frac{M}{2(2\pi)^6}\int d^3k_1\; d^3k_3\; \tilde{v}_{pk_1}(-\vec{k}_1-\vec{k}_3)\left[\delta(\vec{k}-\vec{k}_3)+\delta(2\vec{k}_1-\vec{k}+\vec{k}_3)\right]\psi_p(\vec{k}_3)$$

$$+ \frac{M}{2(2\pi)^6}\int d^3k_1\; d^3k_3\; \tilde{v}_{pk_1}(\vec{k}_1-\vec{k}_3)\left[\delta(2\vec{k}_1+\vec{k}-\vec{k}_3)+\delta(\vec{k}-\vec{k}_3)\right]\psi_p(\vec{k}_3)$$

$$= \frac{M}{2(2\pi)^6}\int d^3k_1\; \tilde{v}_{pk_1}(-\vec{k}_1-\vec{k})\,\psi_p(\vec{k}) +$$

$$+ \frac{M}{2(2\pi)^6}\int d^3k_1\; \tilde{v}_{pk_1}(-\vec{k}_1+2\vec{k}_1-\vec{k})\,\psi_p(\vec{k}-2\vec{k}_1)$$

$$+ \frac{M}{2(2\pi)^6} \int d^3k_1 \, \tilde{v}_{pk_1}(\vec{k}_1-2\vec{k}_1-\vec{k})\psi_p(\vec{k}+2\vec{k}_1) +$$

$$+ \frac{M}{2(2\pi)^6} \int d^3k_1 \, \tilde{v}_{pk_1}(\vec{k}_1-\vec{k})\psi_p(\vec{k})$$

$$= \frac{M}{(2\pi)^6} \int d^3q \, \tilde{v}_{pq}(\vec{q}-\vec{k})\psi_p(\vec{k}) + \frac{M}{(2\pi)^6} \int d^3q \, \tilde{v}_{pq}(\vec{q}-\vec{k})\psi_p(\vec{k}-2\vec{q}).$$

(B9)

Change the variable of integration in the rhs of (B9) from $\vec{q}$ to $\vec{q}'$ given by

$$\vec{q}' = \vec{k} - 2\vec{q}.$$

Then it becomes

$$= \frac{M}{8(2\pi)^6} \int d^3q \, \tilde{v}_{p,\frac{1}{2}(k-q)}(\tfrac{1}{2}\vec{k}-\tfrac{1}{2}\vec{q}-\vec{k})\psi_p(\vec{k})$$

$$+ \frac{M}{8(2\pi)^6} \int d^3q \, \tilde{v}_{p,\frac{1}{2}(k-q)}(\tfrac{1}{2}\vec{k}-\tfrac{1}{2}\vec{q}-\vec{k})\psi_p(\vec{q})$$

$$= \frac{M}{8(2\pi)^6} \int d^3q \, \tilde{v}_{p,\frac{1}{2}(\vec{k}-\vec{q})}(\tfrac{1}{2}\vec{k}+\tfrac{1}{2}\vec{q})(\psi_p(\vec{k}) + \psi_p(\vec{q})) \, ,$$

which, on using eqn. (B4), becomes

$$= \frac{M}{8(2\pi)^6} 8(2\pi)^7 \int d^3q \, \tilde{\tilde{v}}\left[\frac{\frac{1}{2}(\vec{k}+\vec{q})^2+\frac{1}{2}(\vec{k}-\vec{q})^2-p^2}{M}; -2\cdot\tfrac{1}{2}(\vec{k}+\vec{q})\right] \times$$

$$\times (\psi_p(\vec{k}) + \psi_p(\vec{q}))$$

$$= 2\pi M \int d^3q \; \tilde{V}\left[\frac{k^2+q^2-2p^2}{2M} \; ; \; \vec{q}+\vec{k}\right] (\psi_p(\vec{q})+\psi_p(\vec{k})) \; , \tag{B10}$$

which is precisely the term that appears in eqn. (IV.28).

APPENDIX C: PARTIAL WAVE DECOMPOSITION

Define, for the sake of brevity, functions $V_p(\vec{q},\vec{k})$ as follows

$$V_p(\vec{q},\vec{k}) = \tilde{V}\left[\frac{q^2+k^2-2p^2}{2M}\,;\ \vec{q}+\vec{k}\right] \tag{C1}$$

Then

$$V_k(\vec{q},\vec{k}) = \tilde{V}\left[\frac{q^2-k^2}{2M}\,;\ \vec{q}+\vec{k}\right] \tag{C2}$$

Then eqn. (IV.28) for $\psi_p(\vec{k})$ can be written as

$$(p^2-k^2)\psi_p(\vec{k}) = 2\pi M \int d^3q\,[V_p(\vec{q},\vec{k}) - V_k(\vec{q},\vec{k})]\psi_p(\vec{k})$$

$$+ 2\pi M \int d^3q\ V_p(\vec{q},\vec{k})\psi_p(\vec{q}) \tag{C3}$$

The wave function (or scattering amplitude, apart from a factor) is decomposed as follows

$$\psi_p(\vec{k}) = \sum_{\ell m} Y_{\ell m}(\hat{p})Y^*_{\ell m}(\hat{k})\psi^{\ell}_p(k) \tag{C4}$$

Similarly,

$$V_p(\vec{q},\vec{k}) = \sum_{\ell m} Y_{\ell m}(\hat{q})Y^*_{\ell m}(\hat{k})V^{\ell}_p(q,k) \tag{C5}$$

because the p-dependence in $V_p(\vec{q},\vec{k})$ is purely radial.

Rewriting eqn. (IV.29) in terms of $V_p(\vec{q},\vec{k})$ i.e.

$$F(p,k) = 2\pi M \int d^3q \; [V_p(\vec{q},\vec{k}) - V_k(\vec{q},\vec{k})] \qquad (C6)$$

it is obvious that F(p,k) is a function of only the magnitudes of $\vec{p}$ and $\vec{k}$. So, substituting (C4) and (C5) into eqn. (C3) gives:

$$[p^2 - k^2 - F(p,k)] \sum_{\ell m} Y_{\ell m}(\hat{p}) Y^*_{\ell m}(\hat{k}) \psi^{\ell}_p(k)$$

$$= 2\pi M \int d^3q \sum_{\ell m} \sum_{\ell' m'} Y_{\ell m}(\hat{q}) Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{p}) Y^*_{\ell' m'}(\hat{q}) V^{\ell}_p(q,k) \psi^{\ell'}_p(q)$$

$$= 2\pi M \int_0^{\infty} q^2 dq \sum_{\ell m} Y_{\ell m}(\hat{p}) Y^*_{\ell m}(\hat{k}) V^{\ell}_p(q,k) \psi^{\ell}_p(q) \qquad (C7)$$

where one has used the orthogonality relation between the spherical harmonics viz.

$$\int d\Omega q \; Y_{\ell m}(\hat{q}) Y^*_{\ell' m'}(\hat{q}) = \delta_{\ell\ell'} \, \delta_{mm'}$$

Eqn. (C7) then reduces to

$$[p^2 - k^2 - F(p,k)]\psi^{\ell}_p(k) = 2\pi M \int_0^{\infty} q^2 dq \; V^{\ell}_p(q,k) \psi^{\ell}_p(q)$$

This is the equation for the ℓ-th partial wave scattering amplitude; one has yet to determine $V^{\ell}_p(q,k)$. - This is done by inverting eqn. (C5), i.e.

$$V_p^{\ell}(q,k) = 2\pi \int_{-1}^{1} \bar{V}_p(\vec{q},\vec{k}) P_{\ell}(\cos \hat{q}\cdot\hat{k})\, d(\cos \hat{q}\cdot\hat{k}) \qquad (C8)$$

where the following relation has been used:

$$\int_{-1}^{1} P_{\ell}(\cos\theta) P_{\ell'}(\cos\theta)\, d(\cos\theta) = \frac{2}{2\ell+1}\,\delta_{\ell\ell'},$$

in the process of inverting (C5). Combining (C1) and (C8) gives

$$V_p^{\ell}(q,k) = 2\pi \int_{-1}^{1} \tilde{\bar{V}}\left[\frac{q^2+k^2-2p^2}{2M}; \vec{q}+\vec{k}\right] P_{\ell}(\cos \hat{q}\cdot\hat{k})\, d(\cos \hat{q}\cdot\hat{k})$$

$$= \frac{q^2}{(2\pi)^3} \int_{-1}^{1} \frac{P_{\ell}(u)\,du}{\left(\frac{q^2+k^2-2p^2}{2M}\right)^2 - (\vec{q}+\vec{k})^2 - m^2}$$

$$= \frac{4M^2q^2}{(2\pi)^3} \int_{-1}^{1} \frac{P_{\ell}(u)\,du}{(q^2+k^2-2p^2)^2 - 4M^2(q^2+k^2+m^2) - 8M^2kqu}$$

which gives

$$V_p^{\ell}(q,k) = \frac{q^2}{(2\pi)^3} \cdot \frac{1}{kq}\, Q_{\ell}\left[\frac{(q^2+k^2-2p^2)^2 - 4M^2(q^2+k^2+m^2)}{8M^2kq}\right] \qquad (C9)$$

So the s-wave equation is

$$[p^2 - k^2 - F(p,k)]\psi_p^o(k) =$$

$$\frac{Mq^2}{2(2\pi)^2 k} \int_0^{\infty} qdq\, \ell n \left| \frac{(q^2+k^2-2p^2)^2 - 4M^2(q-k)^2 - 4M^2m^2}{(q^2+k^2-2p^2)^2 - 4M^2(q+k)^2 - 4M^2m^2} \right| \psi_p^o(q)$$

(C10)

Next, one wants to obtain an equation for the s-wave part of the T-matrix, which is defined by (IV.31), i.e.

$$\psi_p(\vec{k}) = \frac{1}{\sqrt{2}(2\pi)^3}\left[\delta(\vec{p}+\vec{k}) + \delta(\vec{p}-\vec{k})\right] + \frac{M}{\sqrt{2}(2\pi)^3}\frac{T(k,p)+T(-k,p)}{p^2-k^2+i\eta} \tag{C11}$$

In addition to (C4), one has

$$T(\vec{k},\vec{p}) = \sum_{\ell m} Y_{\ell m}(\hat{k}) Y^*_{\ell m}(\hat{p}) T^{\ell}(k,p) \tag{C12}$$

and

$$\delta(\vec{p}\pm\vec{k}) = \frac{1}{k^2}\,\delta(p-k)\sum_{\ell m}(\pm 1)^{\ell} Y_{\ell m}(\hat{p}) Y^*_{\ell m}(\hat{k}) \tag{C13}$$

Substituting (C4), (C12) and (C13) into (C11) gives:

$$\sum_{\ell m} Y_{\ell m}(\hat{p}) Y^*_{\ell m}(\hat{k})\,\psi^{\ell}_p(k) = \frac{1}{\sqrt{2}(2\pi)^3}\frac{\delta(p-k)}{k^2}\sum_{\ell m}\left[1+(-1)^{\ell}\right] Y_{\ell m}(\hat{p}) Y^*_{\ell m}(k)$$

$$+ \frac{M}{\sqrt{2}(2\pi)^3}\sum_{\ell m}\frac{\left[1+(-1)^{\ell}\right] Y_{\ell m}(\hat{p}) Y^*_{\ell m}(\hat{k})}{p^2-k^2+i\eta}\, T^{\ell}(k,p)$$

which in turn gives

$$\psi^{\ell}_p(k) = \frac{1}{\sqrt{2}(2\pi)^3}\left[1+(-1)^{\ell}\right]\frac{\delta(p-k)}{k^2} + \frac{M}{\sqrt{2}(2\pi)^3}\frac{\left[1+(-1)^{\ell}\right]}{p^2-k^2+i\eta}\, T^{\ell}(k,p) \tag{C14}$$

This equation therefore serves as the definition of the component T^{ℓ} in terms of ψ^{ℓ}. Using (C14), one can derive an equation for the partial wave T-matrix, T^{ℓ}; and this is:

$$T^{\ell}(k,p) = 2\pi \frac{p^2-k^2}{p^2-k^2-F(p,k)} V_p^{\ell}(k,p)$$

$$+ 2\pi M \frac{p^2-k^2}{p^2-k^2-f(p,k)} \int_0^{\infty} \frac{q^2 dq}{p^2-q^2+i} V_p^{\ell}(k,q) T^{\ell}(q,p) \qquad \text{(C15)}$$

Eqn. (C15) holds for all even values of ℓ.

With M(p,k) defined as:

$$M(p,k) = \frac{p^2-k^2}{p^2-k^2-F(p,k)}$$

(C15) becomes, for the s-wave:

$$T^{o}(k,p) = \frac{g^2}{2(2\pi)^2} \frac{M(p,k)}{pk} \ln \left| \frac{(k^2-p^2)^2-4M^2(k-p)^2-4M^2m^2}{(k^2-p^2)^2-4M^2(k+p)^2-4M^2m^2} \right|$$

$$+ \frac{Mg^2}{2(2\pi)^2} \frac{M(p,k)}{k} \int_0^{\infty} \frac{q dq}{p^2-q^2+i\eta} \ln \left| \frac{(q^2+k^2-2p^2)^2-4M^2(q-k)^2-4M^2m^2}{(q^2+k^2-2p^2)^2-4M^2(q+k)^2-4M^2m^2} \right| T^{o}(q,p) \qquad \text{(C16)}$$

This is the equation that is discussed in Section IV.

APPENDIX D: ALGEBRAIC AND NUMERICAL METHODS

In eqn. (IV.59), the following integral appears:

$$J = \int_0^\infty dq \, \frac{q^2 V_p(k,q)}{p^2-q^2+i\eta} \, T_o(q,p)$$

$$= \int_0^\infty dq \, \frac{q^2 V_p(k,q)}{p^2-q^2} \, T_o(q,p) - \frac{i\pi p}{2} \, V_p(k,p) T_o(p) \quad \text{(D1)}$$

where $\int_0^\infty$ indicates a principal value integral. With (D1), and the definition of A(k,p) and B(k,p) [from eqns. (IV.56) and (IV.57)], (IV.59) may be rewritten as:

$$T_o(k,p) - \frac{i\pi M}{2} \, pA(k,p) T_o(p) + B(k,p) \int_0^\infty dq \, \frac{q^2 V_p(k,q)}{p^2-q^2} \, T_o(q,p)$$

$$= A(k,p) \quad \text{(D2)}$$

Although, in what follows, discussion is restricted to the scalar theory, it should be kept in mind that the technique as such carries through for the pseudoscalar model just as well.

In solving (D2), p is kept as a parameter; i.e. p is specified, and a solution obtained for a range of values of k. Eqn. (D2) is first cast into matrix form; $T_o(k,p)$ becomes a column vector, with elements labelled

by $T_o(k_j,p)$ where k_j are the values of k at the grid points j. The left hand side of (D2) is then written as a matrix product

$$\sum_i M_p(k_j,q_i)T_o(q_i,p)$$

whereas the right hand side is a column vector $A(k_j,p)$. The technique then amounts to a matrix inversion, i.e. the matrix $M_p(k_j,q_i)$ is inverted.

Three terms go into the elements of the matrix M_p viz. the three terms on the lhs of (D2). The first two are easily obtained, but in order to obtain contributions from the third, the integral itself has to be evaluated. This is done as follows:

First, one rewrites it, suppressing all but the q-dependence, i.e.

$$I = \int_0^\infty \frac{q^2 V_p(k,q)}{p^2-q^2} T_o(q,p) \equiv \int^\infty dq\ \omega(q)T(q)$$

with

$$\omega(q) = \frac{q^2 V_p(k,q)}{p^2-q^2}$$

and $T(q) = T_o(q,p)$

I is then reduced to an integral over a finite range (0,1) by the transformation

$$q = \frac{pt}{1-t} .$$

Then,

$$I = \int_0^1 \frac{p^2 dt}{(1-t)^2} \omega\left(\frac{pt}{1-t}\right) T\left(\frac{pt}{1-t}\right) \qquad (D3)$$

Next, the range of integration is divided into N equal intervals; i.e. the range is broken into the points $0 = t_o, t_1, \ldots t_j, \ldots t_N$ where

$$t_j = \frac{j}{N}$$

The function T is then approximated within each division (t_{j-1}, t_j) by a straight line constrained to pass through the end points $T\left(\frac{pt_{j-1}}{1-t_{j-1}}\right)$ and $T\left(\frac{pt_j}{1-t_j}\right)$. I then becomes

$$I = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \frac{p^2 dt}{(1-t)^2} \omega\left(\frac{pt}{1-t}\right) \Bigg\{ T\left(\frac{pt_{j-1}}{1-t_{j-1}}\right) + \left[T\left(\frac{pt_j}{1-t_j}\right) - T\left(\frac{pt_{j-1}}{1-t_{j-1}}\right) \right] (Nt-j+1) \Bigg\}$$

If one now transforms back to the q-variable,

$$I = \sum_{j=1}^{N} \int_{q_{j-1}}^{q_j} dq\ \omega(q) \left[T_{j-1} + (T_j - T_{j-1}) \left(\frac{Nq}{q+p} - j+1 \right) \right] \qquad (D4)$$

In (D4),

$$q_j = \frac{pj}{N-j}$$

and

$$T_j \equiv T(q_j)$$

This interpolation therefore removes T(q) from the integral, and hence enables the q-integration to be performed. Obviously, the method is only good if one assumes that T(q) is smooth and singularity-free. Without such an assumption, the interpolation is a very bad approximation.

A typical term in the sum in (D4) is

$$I_j = \int_{q_{j-1}}^{q_j} dq\ \omega(q)\ \left\{ T_{j-1} + (T_j - T_{j-1})\left(\frac{Nq}{q+p} - j + 1\right) \right\}$$

$$= T_{j-1}\ R_j(q_{j-1},q_j) - T_j\ R_{j-1}(q_{j-1},q_j) \qquad \text{(D5)}$$

where $R_j(a,b)$ is given by

$$R_j(a,b) = \int_a^b dq\ \omega(q) \left\{ \frac{Np}{q+p} + j - N \right\} \qquad \text{(D6)}$$

The integrals $R_j(a,b)$ can be evaluated analytically as follows: $\omega(q)$ contains the potential, which in the scalar theory is given by (IV.58)

$$V_p(k,q) = \frac{1}{2kq}\ \ell n \left| \frac{(q^2+k^2-2p^2)^2 - 4M^2(q-k)^2 - 4M^2m^2}{(q^2+k^2-2p^2)^2 - 4M^2(q+k)^2 - 4M^2m^2} \right|$$

In principle, the quartics that appear within the logarithm can be factored, and the logarithm can be replaced by sum of logarithms of the types $\ell n|q+a|$ and $\ell n|(q+x)^2+y^2|$. This depends on whether the zeroes of the quartic are real or complex. We shall just restrict ourselves to the complex case, as the real case is just a limit of this.

Therefore, (D6) consists of terms of the type

$$\frac{1}{2k}\int_a^b \frac{q\,dq}{p^2-q^2}\,\ell n|(q+x)^2+y^2|\,\left(\frac{Np}{q+p}+j-N\right) \qquad (D7)$$

(D7) can be further simplified by decomposing $\frac{q}{p^2-q^2}$; so one is left with terms like

$$\frac{1}{4k}\int_a^b \frac{dq}{q\pm p}\,\ell n|(q+x)^2+y^2|\,\left(\frac{Np}{q+p}+j-N\right) \qquad (D8)$$

Part of (D8) can be written in the form of Euler's dilogarithm[33]

$$D(s) = -\int_1^s \frac{\ell n\, t}{t-1}\,dt$$

by appropriate transformations. s can be real or complex depending on whether the zeroes of the quartic are real or not. The remainder of the integral (D8) can be done in closed form. By this breakdown, therefore, integrals

R_j may be reduced to combinations of dilogarithms (of real and complex arguments) and integrals that can be done in closed form. This in turn implies that I_j may be reduced to a similar form.

This whole procedure hinges upon being able to factor the quartics i.e. determining x,y (or a). This is done numerically; the position of the zeroes will of course depend upon the value of p. Our region is below the single pion threshold.

Eqn. (D2) can therefore be written out in matrix form. However, the discussion so far has been for $k \neq 0$. For $k = 0$, the potential is modified drastically, and the integrals no longer are the same. In fact, they become quite a bit simpler, and the analogues to R_j are exactly evaluated. No dilogarithms are required in this case.

The rest of the calculation has to be done numerically. Computer programs were written to:

a) determine the zeroes of the quartic
b) evaluate the dilogarithm functions (for both real and complex arguments)
c) combine the separate terms
d) incorporate the $k = 0$ limit
e) invert the matrix to determine T.

Numerical consistency was checked by performing the matrix inversion with matrices of different sizes. Grid sizes of 15 and 20 were picked i.e. the matrices were 30 × 30 and 40 × 40. The numbers that are obtained are the real and imaginary parts of the half-off-shell T-matrix elements and the real and imaginary parts of the phase shift, at each value of momentum p (or, in, fact, lab. energy $E_p = \frac{2p^2}{M}$).

APPENDIX E: BOUND STATE INTEGRALS

In Section (IV.6), three integrals were constructed, viz. I_k, I_p, and I_s. The first of these can be evaluated in closed form. I_p and I_s are far more complicated, and this appendix outlines the method of handling them. The integrals are

$$I_s = - A^2 \int_0^\infty \frac{k^2 dk}{(k^2 + \alpha^2)^4} F_b(k)$$

$$I_p = - A^2 \frac{Mg^2}{2(2\pi)^2} \int_0^\infty \frac{kdk}{(k^2+\alpha^2)^2} \int_0^\infty \frac{qdq}{(q^2+\alpha^2)^2} \times$$

$$\times \ln \left| \frac{(q^2+k^2+2ME)^2 - 4M^2(q-k)^2 - 4M^2m^2}{(q^2+k^2+2ME)^2 - 4M^2(q+k)^2 - 4M^2m^2} \right|$$

With the definition of $F_b(k)$, I_s may be written as

$$I_s = - \frac{A^2}{M^3} \frac{g^2}{2(2\pi)^2} (I_1 - I_2) \qquad \text{(E1)}$$

with

$$I_1 = \int_0^\infty \frac{kdk}{(k^2+\alpha^2)^4} \int_0^\infty qdq \ln \left| \frac{(q^2+k^2+\beta^2)^2 - 4(q-k)^2 - 4\lambda^2}{(q^2+k^2+\beta^2)^2 - 4(q+k)^2 - 4\lambda^2} \right| \qquad \text{(E2)}$$

$$I_2 = \int_0^\infty \frac{kdk}{(k^2+\alpha^2)^4} \int_0^\infty qdq \ln \left| \frac{(q^2-k^2)^2 - 4(q-k)^2 - 4\lambda^2}{(q^2-k^2)^2 - 4(q+k)^2 - 4\lambda^2} \right| \qquad \text{(E3)}$$

In arriving at (E2) and (E3), transformation into dimensionless

units has been made i.e.

$$(q,k) \rightarrow (Mq,Mk)$$

Also, μ, β and λ are defined by:

$$\mu = \frac{\alpha}{M}$$

$$\beta = \frac{2E}{M}$$

$$\lambda = \frac{m}{M}$$

For the time being, we shall confine ourselves to I_1. The k-integral in I_1 (S_1, say) may be integrated by parts to give

$$S_1 = \frac{1}{12}\int_0^\infty \frac{dk}{(k^2+\mu^2)^3}\left\{\frac{4k(q^2+k^2+\beta^2)+8(q-k)}{(q^2+k^2+\beta^2)^2-4(q-k)^2-4\lambda^2}\right.$$

$$\left. - \frac{4k(q^2+k^2+\beta^2)-8(q+k)}{(q^2+k^2+\beta^2)^2-4(q+k)^2-4\lambda^2}\right\}$$

when S_1 is re-inserted into I_1, and on defining new variables u,v by

$$k = \frac{1}{\sqrt{2}}(u-v) \qquad q = \frac{1}{\sqrt{2}}(u+v) \qquad \text{(E4)}$$

I_1 becomes

$$I_1 = \frac{1}{3}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; \frac{u+v}{[\frac{1}{2}(u-v)^2+\mu^2]^3} \left\{ \frac{(u-v)(u^2+v^2+\beta^2)+2\sqrt{2}v}{(u^2+v^2+\beta^2)^2-8v^2-4\lambda^2} \right.$$

$$\left. - \frac{(u-v)(u^2+v^2+\beta^2)-2\sqrt{2}u}{(u^2+v^2+\beta^2)^2-8u^2-4\lambda^2} \right\}$$

By interchanging u and v in the second term within the integral, it can be combined with the first one, and eventually I_1 can be written in the following form:

$$I_1 = \frac{2}{3}\frac{\partial^2}{\partial(\mu^2)^2}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; \frac{u+v}{(u-v)^2+2\mu^2} \frac{(u-v)(u^2+v^2+\beta^2)+2\sqrt{2}v}{(u^2+v^2+\beta^2)^2-8v^2-4\lambda^2} \qquad \text{(E5)}$$

Consider the u-integral in (E5); because integration goes from $-\infty$ to $+\infty$, only even powers will survive. Then (E5) may be done by contour integration, as the denominator may be factorised, i.e. first writing I_1 as

$$I_1 = \frac{2}{3}\frac{\partial^2}{\partial(\mu^2)^2}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; \frac{(u+v)[(u+v)^2+2\mu^2]}{[(u-v)^2+2\mu^2][(u+v)^2+2\mu^2]} \times$$

$$\times \frac{(u-v)(u^2+v^2+\beta^2)+2\sqrt{2}v}{(u^2+v^2+\beta^2)^2-8v^2-4\lambda^2}$$

so that the denominator is even, only even powers of u in the numerator have to be considered. The zeroes of the denominator occur at

$$u = \pm v \pm \sqrt{2}i\mu$$

$$u = \pm \left[\pm\sqrt{8v^2+4\lambda^2} - v^2 - \beta^2\right]^{\frac{1}{2}}$$

The structure of the integrand, and also the values of the zeroes show that the result is going to be exceedingly messy. However, it is obtainable in closed form; then the v-integral is evaluated numerically.

I_2 can be treated identically to I_1. Also, I_p which is slightly different in structure, can be evaluated in an analogous fashion. Both I_s and I_p were obtained this way, and have been tabulated in Tables I to III, for certain values of g^2.

APPENDIX F: SELF-ENERGY

The one particle equation is from (V.3),

$$(i\frac{\partial}{\partial x_o} + \frac{\nabla^2}{2M})\langle 0|\psi_{\alpha\alpha'}(x_o,\vec{x})|\vec{p}st\rangle =$$

$$- f_o^2\sigma^i_{\alpha\beta}\vec{\tau}_{\alpha'\beta'}\int dx'_o\, d^3x'\, d^3p'\langle 0|\psi_{\beta\beta'}(x_o,\vec{x})|\vec{p}'s't'\rangle \times$$

$$\times \partial^i\partial^k\bar{\Delta}(x_o-x'_o;\vec{x}-\vec{x}')\langle\vec{p}'s't'|\psi^\dagger_{\gamma\gamma'}(x'_o,\vec{x}')|0\rangle \times$$

$$\times \sigma^k_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\langle 0|\psi_{\delta\delta'}(x'_o,\vec{x}')|\vec{p}st\rangle \qquad \text{(F1)}$$

where s' and t' are summed over on the rhs. Inserting (V.6) into (F1) gives

$$\delta_{\alpha s}\delta_{\alpha' t}\,(i\frac{\partial}{\partial x_o} + \frac{\nabla^2}{2M})u_p(x_o,\vec{x}) =$$

$$f_o^2\sigma^i_{\alpha\beta}\vec{\tau}_{\alpha'\beta'}\int dx'_o\, d^3x'\, d^3p'\,\delta_{\beta s'}\delta_{\beta' t'}u_{p'}(x_o,\vec{x})\,\partial^i\partial^k\bar{\Delta}(x_o-x'_o;\vec{x}-\vec{x}')\times$$

$$\times\,\delta_{\gamma s'}\delta_{\gamma' t'}\,u^*_{p'}(x'_o,\vec{x}')\sigma^k_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\delta_{\delta s}\delta_{\delta' t}u_p(x'_o,\vec{x}') \qquad \text{(F2)}$$

The summation on the rhs of (F2) is

$$\sum_{\substack{\beta\beta' s' t' \\ \gamma\gamma'\delta\delta'}} \sigma^i_{\alpha\beta}\vec{\tau}_{\alpha'\beta'}\delta_{\beta s'}\delta_{\beta' t'}\delta_{\gamma s}\delta_{\gamma' t'}\sigma^k_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\delta_{\delta s}\delta_{\delta' t}$$

$$= \sum_{s't'}\sigma^i_{\alpha s'}\vec{\tau}_{\alpha' t'}\sigma^k_{s' s}\vec{\tau}_{t' t}$$

The remaining sums can be done by including the derivatives $\partial^i\partial^k$ i.e.

$$\partial^i\partial^k \sum_{s't'} \sigma^i_{\alpha s'} \vec{\tau}_{\alpha' t'} \sigma^k_{s's} \vec{\tau}_{t't} = \sum_{s'} (\vec{\sigma}\cdot\vec{\nabla})_{\alpha s'} (\vec{\sigma}\cdot\vec{\nabla})_{s's} \sum_{t'} \vec{\tau}_{\alpha' t'} \cdot \vec{\tau}_{t't}$$

$$= \nabla^2 \delta_{\alpha s} \cdot 3\delta_{\alpha' t}$$

(F2) therefore becomes:

$$\delta_{\alpha s}\delta_{\alpha' t}\left(i\frac{\partial}{\partial x_o} + \frac{\nabla^2}{2M}\right) u_p(x_o,\vec{x}) = 3f_o^2\, \delta_{\alpha s}\delta_{\alpha' t} \int dx_o' d^3x' d^3p' u_{p'}(x_o,\vec{x})$$

$$\nabla_x^2 \bar{\Delta}(x_o - x_o'; \vec{x}-\vec{x}') u^*_{p'}(x_o',\vec{x}') u_p(x_o',\vec{x}')$$

which in turn gives

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla^2}{2M}\right) u_p(x_o,\vec{x}) =$$

$$3f_o^2 \int dx_o'\, d^3x'\, d^3p'\, u_{p'}(x_o,\vec{x}) \nabla_x^2 \bar{\Delta}(x_o - x_o'; \vec{x}-\vec{x}') u^*_{p'}(x_o',\vec{x}') u_p(x_o',\vec{x}') \qquad \text{(F3)}$$

which is precisely eqn. (V.9). If, for $u_p(x_o,\vec{x})$, one uses free solutions, one is led to:

$$\left(E_p - \frac{p^2}{2M}\right) e^{-iE_p x_o} e^{i\vec{p}\cdot\vec{x}} = \frac{3f_o^2}{(2\pi)^3} \int dx_o'\, d^3x'\, d^3p' \times$$

$$\times\, e^{-iE_{p'} x_o} e^{i\vec{p}\cdot\vec{x}} \nabla_x^2 \bar{\Delta}(x_o - x_o'; \vec{x}-\vec{x}') \times$$

$$\times\, e^{i(E_{p'} - E_p)x_o'} e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}'}$$

Therefore,

$$(E_p - \frac{p^2}{2M}) = \frac{3f_o^2}{(2\pi)^3} \int dx_o' \, d^3x' \, d^3p' \, e^{-i(E_p,-E_p)x_o} e^{i(\vec{p}'-\vec{p})\cdot\vec{x}} \times$$

$$\nabla_x^2 \Big\{ \bar{\Delta}(x_o-x_o';\vec{x}-\vec{x}') e^{-i(E_p,-E_p)(x_o-x_o')} e^{i(\vec{p}'-\vec{p})\cdot(\vec{x}-\vec{x}')} \times$$

$$\times e^{i(E_p,-E_p)x_o} e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}} \Big\} \qquad \text{(F4)}$$

But $\int dx_o \, d^3x' \, \bar{\Delta}(x_o-x_o';\vec{x}-\vec{x}') e^{-i(E_p,-E_p)(x_o-x_o')} e^{i(\vec{p}'-\vec{p})\cdot(\vec{x}-\vec{x}')}$

$$= \int dy_o \, d^3y \, \bar{\Delta}(y_o,\vec{y}) e^{-i(E_p,-E_p)y_o} e^{i(\vec{p}'-\vec{p})\cdot\vec{y}}$$

Therefore (F4) becomes

$$E_p - \frac{p^2}{2M} = \frac{3f_o^2}{(2\pi)^3} \int dy_o \, d^3y \, d^3p' \, e^{-i(E_p,-E_p)x_o} e^{i(\vec{p}-\vec{p})\cdot\vec{x}} \times$$

$$\times \bar{\Delta}(y_o,\vec{y}) \, e^{-i(E_p,-E_p)y_o} e^{i(\vec{p}'-\vec{p})\cdot\vec{y}} \times$$

$$\times \nabla_x^2 \left\{ e^{i(E_p,-E_p)x_o} e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}} \right\}$$

$$= -\frac{3f_o^2}{(2\pi)^3} \int dy_o \, d^3y \, d^3p' (\vec{p}'-\vec{p})^2 \bar{\Delta}(y_o,\vec{y}) e^{-i(E_p,-E_p)y_o} e^{i(\vec{p}'-\vec{p})\cdot\vec{y}}$$

Defining

$$W_p = \int d^3p' \, \tilde{\tilde{W}}[E_p,-E_p;\, \vec{p}'-\vec{p}]$$

and

$$\tilde{\tilde{W}}(\omega,k) = -\frac{f_o^2}{(2\pi)^4} k^2 \int dy_o \, d^3y \bar{\Delta}(y_o,\vec{y}) e^{-i\omega y_o} e^{i\vec{k}\cdot\vec{y}}$$

one gets that

$$E_p - \frac{p^2}{2M} = 3(2\pi)W_p \qquad \text{(F5)}$$

We can go a little further, and examine more closely, the effect of self-energy on the two-particle system. If one looks at the equation for any channel, e.g. the (0,1) channel (eqn. (V.43)), the self-energy is seen to be present in the term

$$\frac{p^2-k^2}{p^2-k^2+G(p,k)} \qquad \text{(F6)}$$

In fact, G(p,k) contains the self-energy. We wish to examine (F6) in the zero momentum and infinite momentum limits. Dividing (F6) through by p^2-k^2, we get

$$R(p^2,k^2) = \frac{1}{1 - \frac{G(p,k)}{p^2-k^2}} \qquad \text{(F7)}$$

we want the limit of $R(p^2,k^2)$ when

a) $p^2,k^2 \to 0$

b) $p^2,k^2 \to \infty$

Case (a):

$$F(p,o) = \frac{12M^3 f_o^2}{(2\pi)^2} \int_{\infty}^{\infty} q^4 dq \left(\frac{1}{q^4 - 4(M^2+p^2)q^2 - 4(M^2m^2 - p^4)} \right.$$

$$\left. - \frac{1}{q^4 - 4M^2q^2 - 4M^2m^2} \right) \qquad \text{(F8)}$$

The zeroes of the first quartic in eqn. (F8) are:

$$q^2 = \pm\, \omega_{\pm}^2(p) = 2(M^2 + p^2) \pm 2\sqrt{M^4 + M^2m^2 + 2M^2p^2} \quad ,$$

and therefore, closing the contour in the top half plane, the only zero to contribute will be

$$q = i\omega_-(p)$$

For the second quartic, the zeroes are

$$q^2 = \pm\, \alpha_{\pm}^2 = 2M^2 \pm 2\sqrt{M^4 + M^2m^2}$$

and the only contribution comes from

$$q = i\alpha_-$$

Then, (F8) can be integrated, and gives

$$G(p,o) = -\frac{12M^3 f_o^2}{2\pi}\left[\frac{(M-\frac{m}{M}\frac{p^2}{2M})^3}{4M^2(1+\frac{p^2}{M^2})} - \frac{m^3}{4M^2}\right]$$

which, to order p^2, becomes

$$G(p,o) = \frac{3f_o^2}{2\pi}\frac{m^3}{M} \times \frac{5}{2}\, p^2$$

i.e.

$$\frac{G(p,o)}{p^2} = \frac{15}{4\pi}\, f_o^2\, m^2\, \frac{m}{M}$$

and, eventually

$$R(o,o) = \frac{1}{1 - \frac{15}{4\pi}\frac{m}{M}\,(f_o m)^2} \qquad \text{(F9)}$$

This gives the zero-energy renormalisation. Using the value of f_o^2 i.e.

$$(f_o m)^2 = 4\pi \times 0.08 \ ,$$

F(9) becomes

$$R(o,o) = \frac{1}{1 - 1.2\,\frac{m}{M}} \qquad \text{(F10)}$$

which by virtue of the value of $\frac{m}{M}$, is not appreciably different from 1.

Case (b):

In a similar manner, the infinite limit renormalisation may be obtained. In this case, however, it can be shown that G(p,k) is zero, because the zeroes involved are real, and hence do not contribute to the principal value integral, i.e.

$$\lim_{\substack{p\to\infty \\ k\to\infty}} G(p,k) \equiv 0$$

There is, therefore, no renormalisation in the infinite limit. For large momenta, it is apparent that the approximation G(p,k) = 0 is reasonably good. This approximation has been made in obtaining the Born term (Sect. V.4).

In passing, it should be noted that an analogous calculation may be performed for the scalar meson theory, and again it can be shown that the zero energy renormalisation is small (for the prescribed value of the coupling constant); also it goes to zero in the infinite momentum limit.

APPENDIX G: TWO-NUCLEON SCATTERING STATE EQUATIONS

The field eqn. (V.3) may be relabelled as follows:

$$(i\frac{\partial}{\partial x_o} + \frac{\nabla_z^2}{2M})\psi_{\nu\nu',}(x_o,\vec{z}) = f_o^2\,\sigma^k_{\nu\beta}\vec{\tau}_{\nu',\beta'}\int dx_o' d^3x' \psi_{\beta\beta',}(x_o,\vec{z}) \times$$

$$\times\, \partial^z_k \partial^z_j\, \bar{\Delta}(x_o - x_o'; \vec{z}-\vec{x}')\psi^\dagger_{\gamma\gamma}(x_o',\vec{x}')\sigma^j_{\gamma\delta}\vec{\tau}_{\gamma',\delta'}\psi_{\delta\delta',}(x_o',\vec{x}') \qquad (G1)$$

Taking the expectation value of (G1) between $\langle\vec{q}s_3't_3'|$ and $|\vec{p}_1\vec{p}_2 STs_3t_3\rangle$ gives:

$$(i\frac{\partial}{\partial x_o} + \frac{\nabla_z^2}{2M})\langle\vec{q}s_3't_3'|\psi_{\nu\nu',}(x_o,\vec{z})|\vec{p}_1\vec{p}_2 STs_3t_3\rangle$$

$$= f_o^2\sigma^j_{\nu\beta}\vec{\tau}_{\nu',\beta'}\int dx_o' d^3x' d^3k_1 d^3k_2 d^3k_3 \langle\vec{q}s_3't_3'|\psi_{\beta\beta',}(x_o,\vec{z})|\vec{k}_1\vec{k}_2 S'T's_3''t_3''\rangle \times$$

$$\times\, \partial^z_k\partial^z_j\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\langle\vec{k}_1\vec{k}_2 S'T's_3''t_3''|\psi^\dagger_{\gamma\gamma'}(x_o',x')|\vec{k}_3 s_3'''t_3'''\rangle \times$$

$$\times\, \sigma^k_{\gamma\delta}\vec{\tau}_{\gamma',\delta'}\langle\vec{k}_3 s_3'''t_3'''|\psi_{\delta\delta',}(x_o',x')|\vec{p}_1\vec{p}_2 STs_3t_3\rangle$$

$$+ f_o^2\sigma^j_{\nu\beta}\vec{\tau}_{\nu',\beta'}\int dx_o' d^3x' d^3k_1 d^3k_2 \langle\vec{q}s_3't_3'|\psi_{\beta\beta',}(x_o,z)|\vec{k}_1 ES'T's_3''t_3''\rangle \times$$

$$\times\, \partial^z_k\partial^z_k\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\langle\vec{k}_1 ES'T's_3''t_3''|\psi^\dagger_{\gamma\gamma',}(x_o',\vec{x}')|\vec{k}_2 s_3'''t_3'''\rangle \times$$

$$\times\, \sigma^k_{\gamma\delta}\vec{\tau}_{\gamma',\delta'}\langle\vec{k}_2 s_3'''t_3'''|\psi_{\delta\delta',}(x_o',\vec{x}')|\vec{p}_1\vec{p}_2 STs_3t_3\rangle \qquad (G2)$$

At this point, consider the following:

$$\int (i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M} + \frac{\nabla_z^2}{2M})\{u_q(x_o,\vec{x})\langle\vec{q}s_3't_3'|\psi_{\nu\nu},(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle\}d^3q$$

$$= \int d^3q\, u_q(x_o,\vec{x})(i\frac{\partial}{\partial x_o} + \frac{\nabla_z^2}{2M})\langle\vec{q}s_3't_3'|\psi_{\nu\nu},(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle$$

$$+ \int d^3q\langle\vec{q}s_3't_3'|\psi_{\nu\nu},(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M})u_q(x_o,\vec{x}) \qquad \text{(G3)}$$

But, from (V.7),

$$(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M})u_q(x_o,\vec{x}) = (E_q - \frac{q^2}{2M})u_q(x_o,\vec{x})$$

$$= 3(2\pi)W_q u_q(x_o,\vec{x}) \quad \text{from (V.10)}$$

Therefore (G3) becomes

$$\int d^3q\, u_q(x_o,\vec{x})(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M})\langle\vec{q}s_3't_3'|\psi_{\nu\nu},(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle$$

$$= -3(2\pi)\int d^3q\langle\vec{q}s_3't_3'|\psi_{\nu\nu},(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle W_q u_q(x_o,\vec{x})$$

$$+ (i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M} + \frac{\nabla_z^2}{2M})\int d^3q\, u_q(x_o,\vec{x})\langle\vec{q}s't'|\psi_{\nu\nu},(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle \qquad \text{(G4)}$$

Now multiply (G2) by $u_q(x_o,\vec{x})$ and integrate over $\vec{q}$; this gives:

$$\int d^3q\ u_q(x_o,\vec{x})\,(i\frac{\partial}{\partial x_o} + \frac{\nabla_z^2}{2M})\langle \vec{q}s_3't_3'|\psi_{\nu\nu'}(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle$$

$$= (i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2}{2M} + \frac{\nabla_z^2}{2M}) \int d^3q\ u_q(x_o,\vec{x})\langle \vec{q}s_3't_3'|\psi_{\nu\nu'}(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle$$

$$- 3(2\pi) \int d^3q\ W_q\ u_q(x_o,\vec{x})\langle \vec{q}s_3't_3'|\psi_{\nu\nu'}(x_o,\vec{z})|\vec{p}_1\vec{p}_2STs_3t_3\rangle$$

$$= f_o^2\sigma^k_{\nu\beta}\vec{\tau}_{\nu'\beta'} \int dx_o'\ d^3x'\ d^3k_1\ d^3k_2\ d^3k_3\ d^3q\ u_q(x_o,\vec{x})\ \times$$

$$\times \langle \vec{q}s_3't_3'|\psi_{\beta\beta'}(x_o,\vec{z})|\vec{k}_1\vec{k}_2S'T's_3''t_3''\rangle \partial_k^z\partial_j^z\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\ \times$$

$$\times \langle \vec{k}_1\vec{k}_2S'T's_3''t_3''|\psi^\dagger_{\gamma\gamma'}(x_o',\vec{x}')|\vec{k}_3s_3'''t_3'''\rangle\sigma^j_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\ \times$$

$$\times \langle \vec{k}_3s_3'''t_3'''|\psi_{\delta\delta'}(x_o',\vec{x}')|\vec{p}_1\vec{p}_2STs_3t_3\rangle$$

$$+ f_o^2\sigma^k_{\nu\beta}\vec{\tau}_{\nu'\beta'} \int dx_o'\ d^3x'\ d^3k_1\ d^3k_2\ u_q(x_o,\vec{x})\ \times$$

$$\times \langle \vec{q}s_3't_3'|\psi_{\beta\beta'}(x_o,\vec{z})|\vec{k}_1ES'T's_3''t_3''\rangle \partial_k^z\partial_j^z\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\ \times$$

$$\times \langle \vec{k}_1ES'T's_3''t_3''|\psi^\dagger_{\gamma\gamma'}(x_o',\vec{x}')|\vec{k}_2s_3'''t_3'''\rangle\sigma^j_{\gamma\delta}\vec{\tau}_{\gamma'\delta'}\ \times$$

$$\times \langle \vec{k}_2s_3'''t_3'''|\psi_{\delta\delta'}(x_o',\vec{x}')|\vec{p}_1\vec{p}_2STs_3t_3\rangle \qquad \text{(G5)}$$

Now one needs the following:

$$W_q\ u_q(x_o,\vec{x}) = W_q\ Ae^{-i(E_qx_o-\vec{q}\cdot\vec{x})}$$

$$= \int d^3k\, W_k\, Ae^{-i(E_q x_o - \vec{k}\cdot\vec{x})}\, \delta(\vec{k}-\vec{q})$$

$$= \frac{1}{(2\pi)^3} \int d^3k\, d^3x'\, W_k\, Ae^{-i(E_q x_o - \vec{k}\cdot\vec{x})}\, e^{-i(\vec{k}-\vec{q})\cdot\vec{x}'}$$

$$= \frac{1}{(2\pi)^3} \int d^3x' \int d^3k\, W_k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, Ae^{-i(E_q x_o - \vec{q}\cdot\vec{x}')},$$

which, by (V.13)

$$= \int d^3x'\, W_o(\vec{x}-\vec{x}')\, u_q(x_o,\vec{x}') \qquad \text{(G6)}$$

Also,

$$\int d^3z\, u_k(x_o,\vec{z})\, u^*_{k'}(x_o,\vec{z}) = \delta(\vec{k}-\vec{k}')$$

Therefore,

$$|\vec{k}_3 s_3''' t_3'''\rangle\langle \vec{k}_3 s_3''' t_3'''| \equiv |\vec{k}_3 s_3''' t_3'''\rangle \int d^3k' \delta(\vec{k}_3-\vec{k}')\langle \vec{k}' s_3'' t_3''|$$

$$= |\vec{k}_3 s_3''' t_3''\rangle \int d^3k' d^3z' u_{k_3}(x_o',\vec{z}') u^*_{k'}(x_o',\vec{z}')\langle \vec{k}' s_3''' t_3'''|.$$

In this manner, one can introduce the u_k's into eqn. (G5). This leads to

$$\left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right) \int d^3q\, u_q(x_o,\vec{x})\langle \vec{q} s_3' t_3'|\psi_{\nu\nu'}(x_o,\vec{z})|\vec{p}_1\vec{p}_2 STs_3t_3\rangle$$

$$= 3(2\pi)\int d^3x' W_o(\vec{x}-\vec{x}')\int d^3q u_q(x_o,\vec{x}')\langle \vec{q} s't'|\psi_{\nu\nu'}(x_o,\vec{z})|\vec{p}_1\vec{p}_2 STs_3t_3\rangle$$

$$+ f_o^2 \sigma^k_{\nu\beta} \vec{\tau}_{\nu',\beta'} \sigma^j_{\gamma\delta} \vec{\tau}_{\gamma',\delta'} \int d^3q u_q(x_o,\vec{x}) \langle \vec{q} s_3' t_3' | \psi_{\beta\beta'}(x_o,\vec{z}) | \vec{k}_1 \vec{k}_2 S'T' s_3'' t_3'' \rangle$$

$$\times \partial^z_k \partial^z_j \bar{\Delta}(x_o - x_o'; \vec{z} - \vec{x}') \int d^3k_3 \langle \vec{k}_1 \vec{k}_2 S'T' s_3'' t_3'' | \psi^\dagger_{\gamma\gamma'}(x_o',\vec{x}') | \vec{k}_3 s_3''' t_3''' \rangle \times$$

$$\times u^*_{k_3}(x_o',z') \int d^3k' u_{k'}(x_o',z') \langle \vec{k}' s''' t''' | \psi_{\delta\delta'}(x_o',\vec{x}') | \vec{p}_1 \vec{p}_2 ST s_3 t_3 \rangle \times$$

$$\times d^3x' \, dx_o' \, d^3k_1 \, d^3k_2$$

$$+ \text{the corresponding bound state contribution.} \qquad \text{(G7)}$$

Finally, using (V.15) i.e.

$$\int d^3q \; u_q(x_o,\vec{x}) \langle \vec{q} s_3' t_3' | \psi_{\beta\beta'}(x_o,\vec{z}) | \vec{k}_1 \vec{k}_2 S'T' s_3'' t_3'' \rangle$$

$$= \langle s_3' \beta | S' s_3'' \rangle \langle t_3' \beta' | T' t_3'' \rangle \; F^{S'T'}_{k_1 k_2}(t,\vec{z},\vec{x}) \; ,$$

one gets the following equation from (G7):

$$\langle s_3' \nu | S s_3 \rangle \langle t_3' \nu' | T t_3 \rangle \left(i \frac{\partial}{\partial x_o} + \frac{\nabla^2_x + \nabla^2_z}{2M} \right) F^{ST}_{p_1 p_2}(x_o,\vec{z},\vec{x})$$

$$= 3(2\pi) \langle s_3' \nu | S s_3 \rangle \langle t_3' \nu' | T t_3 \rangle \int d^3x' \; W_o(\vec{x} - \vec{x}') F^{ST}_{p_1 p_2}(x_o,\vec{z},\vec{x}')$$

$$+ f_o^2 \sigma^k_{\nu\beta} \vec{\tau}_{\nu',\beta'} \sigma^j_{\gamma\delta} \vec{\tau}_{\gamma',\delta'} \int \langle s_3' \beta | S' s_3'' \rangle \langle t_3' \beta' | T' t_3'' \rangle \langle S' s_3'' | s_3''' \gamma \rangle \times$$

$$\times \langle T' t_3'' | t_3''' \gamma' \rangle \langle s_3''' \delta | S s_3 \rangle \langle t_3''' \delta' | T t_3 \rangle F^{S'T'}_{k_1 k_2}(x_o,\vec{z},\vec{x}) \partial^z_k \partial^z_j \times$$

$$\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')F^{S'T'*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{ST}_{p_1p_2}(x_o,\vec{x}',\vec{z}')d^3x'dx_o'd^3k_1d^3k_2$$

+ the corresponding bound state contribution. (G8)

On the right hand side of (G8), there is summation over $\beta,\beta',\delta,\delta',S',s_3'',T',t_3'',\gamma,\gamma'$, and t_3'''. A similar equation can be derived in exactly the same fashion for the bound state wavefunction.

APPENDIX H: ISOSPIN DEPENDENCE

Before one can evaluate the combinations of Clebsch-Gordan coefficients that appear in (V.16), one needs the following coefficient:

$$\begin{aligned}\langle ab|Tt_3\rangle &= \delta_{T0}\,\delta_{t_3 0}\,\frac{1}{\sqrt{2}}\,[\delta_{a+}\,\delta_{b-} - \delta_{a-}\,\delta_{b+}] \\ &+ \delta_{T1}\{\delta_{t_3 1}\delta_{a+}\delta_{b+} + \delta_{t_3 0}\,\frac{1}{\sqrt{2}}\,[\delta_{a+}\delta_{b-} + \delta_{a-}\delta_{b+}] \\ &+ \delta_{t_3 -1}\delta_{a-}\delta_{b-}\} \qquad \text{(H1)}\end{aligned}$$

where the ± stand for ±1/2, since one is considering two isospin 1/2 particles. With (H1), one can evaluate some of the sums in (V.16). First, the sums over t_3'' and t_3''' are:

$$\begin{aligned}&\sum_{t_3'' t_3'''} \langle t_3'\beta'|T' t_3''\rangle\langle T' t_3''|t_3'''\gamma'\rangle\langle t_3'''\delta'|Tt_3\rangle \\ &= \sum_{t_3'' t_3'''} \{\delta_{T'0}\,t_3''0\,\tfrac{1}{2}[\delta_{t_3'+}\delta_{\beta'-} - \delta_{t_3'-}\delta_{\beta'+}][\delta_{t_3'''+}\delta_{\gamma'-} - \delta_{t_3'''-}\delta_{\gamma'+}] \\ &+ \delta_{T'1}\delta_{t_3''1}\delta_{t_3'+}\delta_{\beta'+}\delta_{t_3'''+}\delta_{\gamma'+} + \delta_{T'1}\delta_{t_3''0}\,\tfrac{1}{2}[\delta_{t_3'+}\delta_{\beta'-} + \delta_{t'-}\delta_{\beta'+}]\times \\ &\times [\delta_{t_3'''+}\delta_{\gamma'-} + \delta_{t_3'''-}\delta_{\gamma'+}] + \delta_{T'1}\delta_{t_3'''-1}\delta_{t_3'-}\delta_{t_3'-}\delta_{\beta'-}\delta_{t_3'''-}\delta_{\gamma'-}\} \times \\ &\times \langle t_3'''\delta'|Tt_3\rangle\end{aligned}$$

$$= \sum_{t_3'''} \{\delta_{T'0} \tfrac{1}{2}[\delta_{t_3'+}\delta_{\beta'-} - \delta_{t_3'-}\delta_{\beta'+}][\delta_{t_3'''+}\delta_{\gamma'-} - \delta_{t_3'''-}\delta_{\gamma'+}]$$

$$+ \delta_{T'1}\Big(\delta_{t_3'+}\delta_{\beta'+}\delta_{t_3'''+}\delta_{\gamma'+} + \tfrac{1}{2}[\delta_{t_3'+}\delta_{\beta'-}+\delta_{t_3'-}\delta_{\beta'+}] \times$$

$$\times [\delta_{t_3'''+}\delta_{\gamma'-} + \delta_{t_3'''-}\delta_{\gamma'+}] + \delta_{t_3'-}\delta_{\beta'-}\delta_{t_3'''-}\delta_{\gamma'-}\Big)\}\langle t_3'''\delta'|Tt_3\rangle$$

$$= \tfrac{1}{2}\delta_{T'0}[\delta_{t_3'+}\delta_{\beta'-} - \delta_{t_3'-}\delta_{\beta'+}][\langle +\delta'|Tt_3\rangle\delta_{\gamma'-} - \langle -\delta'|Tt_3\rangle\delta_{\gamma'+}]$$

$$+ \delta_{T'1}\delta_{t_3'+}\delta_{\beta'+}\langle +\delta'|Tt_3\rangle\delta_{\gamma'+} + \delta_{T'1}\delta_{t_3'-}\delta_{\beta'-}\langle -\delta'|Tt_3\rangle\delta_{\gamma'-}$$

$$+ \tfrac{1}{2}\delta_{T'1}[\delta_{t_3'+}\delta_{\beta'-} + \delta_{t_3'-}\delta_{\beta'+}][\langle +\delta'|Tt_3\rangle\delta_{\gamma'-} + \langle -\delta'|Tt_3\rangle\delta_{\gamma'+}] \qquad \text{(H2)}$$

One also needs the following:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad \text{(H3)}$$

Since τ_1, τ_2, τ_3 form a vector , one can modify the notation and write

$$\vec{\tau}_{++} = (0, 0, 1) \qquad \vec{\tau}_{-+} = (1, i, 0)$$
$$\vec{\tau}_{+-} = (1, -i, 0) \qquad \vec{\tau}_{--} = (0, 0, -1) \qquad \text{(H4)}$$

where the labelling for each τ_i is the following:

$$\begin{pmatrix} ++ & +- \\ -+ & -- \end{pmatrix}$$

From (H4), it is trivial to obtain the following products:

$$\vec{\tau}_{\nu+} \cdot (\vec{\tau}_{++}, \vec{\tau}_{+-}, \vec{\tau}_{-+}, \vec{\tau}_{--}) = (\delta_{\nu+}, 2\delta_{\nu-}, 0, -\delta_{\nu+})$$

$$\vec{\tau}_{\nu-} \cdot (\vec{\tau}_{++}, \vec{\tau}_{+-}, \vec{\tau}_{-+}, \vec{\tau}_{--}) = (-\delta_{\nu-}, 0, 2\delta_{\nu+}, \delta_{\nu-})$$

(H5)

Eqns. (H2) to (H5) permit one to explicitly handle the isospin dependence of (V.16); one has to consider the cases T = 0 and T = 1 separately.

a) Total Isospin T = 0.

One is interested in <u>only</u> the isospin part of eqn. (V.16); so everything else will be ignored (for the time being, at least). Symbolically, therefore, the left hand side of (V.16) may be represented by

$$\langle t_3' \nu' | 00 \rangle F^{S0} \qquad \text{(H6)}$$

The right hand side has two terms: the first term is of the same form as (H6); the second is represented by

$$\sum_{\substack{\beta'\gamma'\delta' \\ T't_3''t_3'''}} \vec{\tau}_{\nu'\beta'} \vec{\tau}_{\gamma'\delta'} \langle t_3'\beta' | T't_3'' \rangle \langle T't_3'' | t_3'''\gamma' \rangle \langle t_3'''\delta' | 00 \rangle \times F^{S'T'} F^{S'T'*} F^{S0} \qquad \text{(H7)}$$

which, upon using (H2) becomes

$$= \sum_{\substack{\beta'\gamma' \\ S'T'}} \vec{\tau}_{\nu'\beta'} \vec{\tau}_{\gamma'\beta'} \{\tfrac{1}{2}\, \delta_{T'0}[\delta_{t_3'+}\delta_{\beta'-} - \delta_{t_3'-}\delta_{\beta'+}] \times$$

$$\times \quad [\langle +\delta'|00\rangle \delta_{\gamma'-} - \langle -\delta'|00\rangle \delta_{\gamma'+}]$$

$$+ \delta_{T'1}\delta_{t_3'+}\delta_{\beta'+} \langle +\delta'|00\rangle \delta_{\gamma'+} + \delta_{T'1}\delta_{t_3'-}\delta_{\beta'-}\langle -\delta'|00\rangle \delta_{\gamma'-}$$

$$+ \tfrac{1}{2}\, \delta_{T'1}[\delta_{t_3'+}\delta_{\beta'-} + \delta_{t_3'-}\delta_{\beta'+}][\langle +\delta'|00\rangle \delta_{\gamma'-} + \langle -\delta'|00\rangle \delta_{\gamma'+}]\} \times$$

$$\times F^{S'T'} F^{S'T'*} F^{S0}$$

$$= \frac{3}{\sqrt{2}}\, \delta_{t'+}\delta_{\nu'-} F^{S'1} F^{S'1*} F^{S0} - \frac{3}{\sqrt{2}}\, \delta_{t'-}\delta_{\nu'+} F^{S'1} F^{S'1*} F^{S0}$$

$$= 3 \cdot \frac{1}{\sqrt{2}}\, [\delta_{t'+}\delta_{\nu'-} - \delta_{t'-}\delta_{\nu'+}] F^{S'1} F^{S'1*} F^{S0}$$

$$= 3 \langle t_3'\nu'|00\rangle\, F^{S'1} F^{S'1*} F^{S0} \qquad \text{(H8)}$$

It is obvious from (H8) that the T = 0 state couples only to T = 1 states. Also, (H6) and (H8) show that the lhs and rhs of eqn. (V.16) for T = 0 have the same Clebsch-Gordan coefficient. So the equation for the T = 0 channel (total spin S being unspecified) is:

$$\langle s_3'\nu|Ss_3\rangle \left(i\frac{\partial}{\partial x_o} + \frac{\nabla_x^2 + \nabla_z^2}{2M}\right) F^{S0}_{p_1p_2}(x_o, \vec{z}, \vec{x})$$

$$= 3(2\pi) \langle s_3'\nu|Ss_3\rangle \int d^3x'\, W_o(\vec{x}-\vec{x}')\, F^{S0}_{p_1p_2}(x_o, \vec{z}, \vec{x}')$$

$$+ 3f_o^2\, \sigma^i_{\nu\beta}\, \sigma^k_{\gamma\delta} \int dx_o' d^3x' d^3k_1 d^3k_2 d^3z'\, \langle s_3'\beta|S's_3''\rangle \langle S's_3''|s_3'''\gamma\rangle$$

$$\times \langle s_3'''\delta|Ss_3\rangle F^{S'1}_{k_1k_2}(x_o,\vec{z},\vec{x})\,\partial^z_k\partial^z_j\,\bar{\Delta}(x_o-x_o';\,\vec{z}-\vec{x}')\,\times$$

$$\times F^{S'1*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')\;F^{S0}_{p_1p_2}(x_o',\vec{x}',\vec{z}')$$

+ the corresponding bound state term. (H9)

b) Total Isospin T = 1.

An analogous treatment to that in (a) can be done here. Since the method is identical, and the algebra much longer than in (a), it will not be reproduced here. Instead, the final equation will just be written down, and is:

$$\langle s_3'\nu|Ss_3\rangle\,(i\frac{\partial}{\partial x_o}+\frac{\nabla^2_x+\nabla^2_z}{2M})F^{S1}_{p_1p_2}(x_o,\vec{z},\vec{x})$$

$$=3(2\pi)\langle s_3'\nu|Ss_3\rangle\int d^3x'\;W_o(\vec{x}-\vec{x}')F^{S1}_{p_1p_2}(x_o,\vec{z},\vec{x}')$$

$$+f_o^2\,\sigma^j_{\nu\beta}\,\sigma^k_{\gamma\beta}\int dx_o'd^3x'd^3z'd^3k_1d^3k_2\langle s_3'\beta|S's_3''\rangle\langle S's_3''|s_3'''\gamma\rangle\,\times$$

$$\times\langle s_3'''\delta|Ss_3\rangle F^{S'0}_{k_1k_2}(x_o,\vec{z},\vec{x})\,\partial^z_k\partial^z_j\,\bar{\Delta}(x_o-x_o';\vec{z}-\vec{x}')\,\times$$

$$\times F^{S'0*}_{k_1k_2}(x_o',\vec{x}',\vec{z}')F^{S1}_{p_1p_2}(x_o',\vec{x}',\vec{z}')$$

$$+2f_o^2\,\sigma^j_{\nu\beta}\,\sigma^k_{\gamma\delta}\int dx_o'\,d^3x'\,d^3z'\,d^3k_1\,d^3k_2\,\times$$

$$\times\langle s_3'\beta|S's_3''\rangle\langle S's_3''|s_3'''\gamma\rangle\langle s_3'''\delta|Ss_3\rangle\,\times$$

$$\times F^{S'1}_{k_1k_2}(x_o,\vec{z},\vec{x})\,\partial^z_k\partial^z_j\,\bar{\Delta}(x_o-x'_o;\,\vec{z}-\vec{x}')F^{S'1*}_{k_1k_2}(x'_o,\vec{x}',\vec{z}')\times$$

$$\times F^{S1}_{p_1p_2}(x'_o,\vec{x}',\vec{z}')$$

+ the corresponding bound state terms (H10)

Unlike the T = 0 case, the T = 1 state couples to <u>both</u> T = 0 and T = 1 intermediate states, with different weights.

APPENDIX I: SPIN DEPENDENCE

The calculation here is straightforward, and is very similar to that in App. H. A relation identical to eqn. (H2) is obtained for the spins. The full expression required is

$$\sum_{\substack{\beta\gamma\delta \\ S's_3''s_3'''}} (\vec{\sigma}\cdot\vec{\nabla})_{\nu\beta}(\vec{\sigma}\cdot\vec{\nabla})_{\gamma\delta}\langle s_3'\beta|S's_3''\rangle\langle S's_3''|s_3'''\gamma\rangle \times$$

$$\times \langle s_3'''\delta|Ss_3\rangle\, F^{S'T'}F^{S'T'*}F^{ST} \qquad \text{(I1)}$$

(I1) is analogous to (H7) for the more general case (i.e. T not always = 0), where the isospin Clebsch-Gordan coefficients are replaced by spin quantities, and in addition

$$\vec{\tau} \rightarrow (\vec{\sigma}\cdot\vec{\nabla})$$

Therefore, as before, (I1) becomes, for S = 0:

$$= \sum_{\delta} (\vec{\sigma}\cdot\vec{\nabla})_{\nu-}(\vec{\sigma}\cdot\vec{\nabla})_{-\delta}\left(\frac{1}{2}\,\delta_{s_3'+}\langle+\delta|00\rangle F^{0T'}F^{0T'*}\right.$$

$$\left. + \frac{1}{2}\,\delta_{s_3'-}\langle+\delta|00\rangle F^{1T'}F^{1T'*} + \delta_{s_3'-}\langle-\delta|00\rangle F^{1T'}F^{1T'*}\right)_{F^{0T}}$$

$$+ \sum_{\delta} (\vec{\sigma}\cdot\vec{\nabla})_{\nu-}(\vec{\sigma}\cdot\vec{\nabla})_{+\delta}\left(-\frac{1}{2}\,\delta_{s_3'+}\langle-\delta|00\rangle F^{0T'}F^{0T'*}\right.$$

$$+ \frac{1}{2}\,\delta_{s_3'+}\,{}^{<-\delta|00>}F^{1T'}F^{1T'*}\Big\}\,F^{0T}$$

$$+ \sum_{\delta}(\vec{\sigma}\cdot\vec{\nabla})_{\nu+}(\vec{\sigma}\cdot\vec{\nabla})_{-\delta}\Big(-\frac{1}{2}\,\delta_{s_3'-}\,{}^{<+\delta|00>}F^{0T'}F^{0T'*}$$

$$+ \frac{1}{2}\,\delta_{s_3'-}\,{}^{<+\delta|00>}F^{1T'}F^{1T'*}\Big)\,F^{0T}$$

$$+ \sum_{\delta}(\vec{\sigma}\cdot\vec{\nabla})_{\nu+}(\vec{\sigma}\cdot\vec{\nabla})_{+\delta}\Big(\frac{1}{2}\,\delta_{s_3'-}\,{}^{<-\delta|00>}F^{0T'}F^{0T'*}$$

$$+ \frac{1}{2}\,\delta_{s_3'-}\,{}^{<-\delta|00>}F^{1T'}F^{1T'*} + \delta_{s_3'+}\,{}^{<+\delta|00>}F^{1T'}F^{1T'*}\Big)_{F^{0T}}$$

(I2)

In order to evaluate (I2) one needs the following:

$$(\vec{\sigma}\cdot\vec{\nabla}) = \begin{pmatrix}\partial_z & \partial_- \\ \partial_+ & -\partial_z\end{pmatrix} \tag{I3}$$

where $\partial_\pm = \partial_x \pm i\partial_y$; from (I3), the following products are really obtained:

$$(\vec{\sigma}\cdot\vec{\nabla})_{\nu+}\cdot\{(\vec{\sigma}\cdot\vec{\nabla})_{++},\ (\vec{\sigma}\cdot\vec{\nabla})_{+-},\ (\vec{\sigma}\cdot\vec{\nabla})_{-+},\ (\vec{\sigma}\cdot\vec{\nabla})_{--}\}$$

$$= \delta_{\nu+}\{\partial_z^2, \partial_z\partial_-, \partial_+\partial_z, -\partial_z^2\} + \delta_{\nu-}\{\partial_+\partial_z, \partial_+\partial_-, \partial_+^2, -\partial_+\partial_z\} \tag{I4}$$

and

$$(\vec{\sigma}\cdot\vec{\nabla})_{\nu-}\cdot\{(\vec{\sigma}\cdot\vec{\nabla})_{++}, (\vec{\sigma}\cdot\vec{\nabla})_{+-}, (\vec{\sigma}\cdot\vec{\nabla})_{-+}, (\vec{\sigma}\cdot\vec{\nabla})_{--}\}$$

$$= \delta_{\nu+}\{\partial_-\partial_z, \partial_-^2, \partial_-\partial_+, -\partial_-\partial_z\} + \delta_{\nu-}\{-\partial_z^2, -\partial_z\partial_-, -\partial_+\partial_z, \partial_z^2\}$$

(I5)

Therefore, (I2) becomes

$$= [\delta_{\nu+}(-\partial_-\partial_z)+\delta_{\nu-}(\partial_z^2)]\left[\frac{1}{2\sqrt{2}}\,\delta_{s_3'+}\,F^{0T'}F^{0T'*}+\frac{1}{2\sqrt{2}}\,\delta_{s_3'+}F^{1T'}F^{1T'*}\right]F^{0T}$$

$$+ [\delta_{\nu+}(\partial_-\partial_+) + \delta_{\nu-}(-\partial_+\partial_z)]\left[-\frac{1}{\sqrt{2}}\,\delta_{s_3'-}F^{1T'}F^{1T*}\right]F^{0T}$$

$$+ [\delta_{\nu+}(\partial_-\partial_z)+\delta_{\nu-}(-\partial_z^2)]\left[\frac{1}{2\sqrt{2}}\delta_{s_3'+}F^{0T'}F^{0T'*}-\frac{1}{2\sqrt{2}}\delta_{s_3'+}F^{1T'}F^{1T'*}\right]F^{0T}$$

$$+ [\delta_{\nu+}(\partial_z^2)+\delta_{\nu-}(\partial_+\partial_z)]\left[-\frac{1}{2\sqrt{2}}\,\delta_{s_3'-}F^{0T'}F^{0T'*}-\frac{1}{2\sqrt{2}}\delta_{s_3'-}F^{1T'}F^{1T'*}\right]F^{0T}$$

$$+ [\delta_{\nu+}(-\partial_z^2)+\delta_{\nu-}(-\partial_+\partial_z)]\left[-\frac{1}{2\sqrt{2}}\delta_{s_3'-}F^{0T'}F^{0T'*}+\frac{1}{2\sqrt{2}}\delta_{s_3'-}F^{1T'}F^{1T'*}\right]F^{0T}$$

$$+ [\delta_{\nu+}(\partial_z\partial_-)+\delta_{\nu-}(\partial_+\partial_-)]\left[\frac{1}{\sqrt{2}}\,\delta_{s_3'+}\,F^{1T'}F^{1T*}\right]F^{0T}$$

$$= \delta_{\nu+}\,\delta_{s_3'-}\left[-\frac{1}{\sqrt{2}}\,\partial_-\partial_+ - \frac{1}{2\sqrt{2}}\,\partial_z^2 - \frac{1}{2\sqrt{2}}\,\partial_z^2\right]F^{1T'}F^{1T'*}F^{0T}$$

$$+ \delta_{\nu-}\,\delta_{s_3'+}\left[\frac{1}{2\sqrt{2}}\,\partial_z^2 + \frac{1}{2\sqrt{2}}\,\partial_z^2 + \frac{1}{\sqrt{2}}\,\partial_-\partial_+\right]F^{1T'}F^{1T*}F^{0T}$$

$$= \frac{1}{\sqrt{2}}\left[\delta_{s_3'+}\,\delta_{\nu-} - \delta_{s_3'-}\,\delta_{\nu+}\right]\nabla^2F^{1T'}F^{1T'*}F^{0T}$$

$$= \langle s_3'\,\nu|00\rangle\,\nabla^2F^{1T'}F^{1T'*}F^{0T} \qquad \text{(I6)}$$

The derivation of the corresponding term for S = 1 is far more laborious, and eventually leads to

$$\frac{1}{2}\,\langle s_3'\nu|\nabla^2 - (\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\vec{\nabla})|1s_3\rangle\,F^{0T'}F^{0T'*}F^{0T}$$

$$+ \frac{1}{2} \langle s_3' \nu | \nabla^2 + (\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\vec{\nabla}) | 1 s_3 \rangle F^{1T'} F^{1T'*} F^{1T} \qquad (I7)$$

This is the term corresponding to (I6); once again, as in the case of isospin, it is apparent from (I6) and (I7) that singlet states (S = 0) couple only to triplet states (S = 1) while triplet states themselves couple to both.

APPENDIX J: ETACR AND COMPLETENESS

Eqns. (V.5) for the ETACR leads to a completeness relation for two-nucleon states in the following manner:

$$\langle \vec{p}st | [\psi_{\alpha\alpha'}(x_o,\vec{x}), \psi^{\dagger}_{\beta\beta'}(x_o,\vec{x}')]_+ | \vec{p}'s't' \rangle =$$

$$\delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{ss'}\delta_{tt'}\delta(\vec{p}-\vec{p}')\delta(\vec{x}-\vec{x}') \qquad (J1)$$

i.e.

$$\sum_{\substack{Ss_3 \\ Tt_3}} \int d^3q_1\, d^3q_2 \langle \vec{p}st | \psi_{\alpha\alpha'}(x_o,\vec{x}) | \vec{q}_1\vec{q}_2 STs_3t_3 \rangle \times$$

$$\times \langle \vec{q}_1\vec{q}_2 STs_3t_3 | \psi^{\dagger}_{\beta\beta'}(x_o,\vec{x}') | \vec{p}'s't' \rangle$$

$$+ \langle \vec{p}st | \psi^{\dagger}_{\beta\beta'}(x_o,\vec{x}') | 0 \rangle \langle 0 | \psi_{\alpha\alpha'}(x_o,\vec{x}) | \vec{p}'s't' \rangle$$

$$= \delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{ss'}\delta_{tt'}\delta(\vec{p}-\vec{p}')\delta(\vec{x}-\vec{x}') \qquad (J2)$$

Multiply through by $u_p(x_o,\vec{z})u^*_{p'}(x_o,\vec{z}')$ (defined in Section V.1) and integrate over d^3p, d^3p'; then

$$\sum_{\substack{Ss_3 \\ Tt_3}} \int d^3q_1\, d^3q_2\, d^3p\, d^3p'\, u_p(x_o,\vec{z}) \langle \vec{p}st | \psi_{\alpha\alpha'}(x_o,\vec{x}) | \vec{q}_1\vec{q}_2 STs_3t_3 \rangle \times$$

$$\times \langle \vec{q}_1\vec{q}_2 STs_3t_3 | \psi^{\dagger}_{\beta\beta'}(x_o,\vec{x}') | \vec{p}'s't' \rangle u^*_{p'}(x_o,\vec{z}')$$

$$+ \int d^3p d^3p'\, u_p(x_o,\vec{z})\delta_{\alpha s}\delta_{\alpha' t'} u^*_{p'}(x_o,\vec{x}) u_{p'}(x_o,\vec{z}')\delta_{\beta s}\delta_{\beta' t} u^*_p(x_o,\vec{x}')$$

$$= \delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{ss'}\delta_{tt'}\int d^3p d^3p'\; u_p(x_o,\vec{z})u^*_{p'}(x_o,\vec{z}')\delta(\vec{p}-\vec{p}')\delta(\vec{x}-\vec{x}')$$

where eqns. (V.6) has been used. Now, with eqn. (V.15), this becomes

$$\sum_{\substack{Ss_3\\Tt_3}} \int d^3q_1 d^3q_2 \langle s\alpha|Ss_3\rangle\langle t\alpha'|Tt_3\rangle F^{ST}_{q_1q_2}(x_o,\vec{x},\vec{z})\langle s'\beta|Ss_3\rangle\langle t'\beta'|Tt_3\rangle \times$$

$$\times F^{ST*}_{q_1q_2}(x_o,\vec{x}',\vec{z}')$$

$$+ \delta_{\alpha s'}\delta_{\beta s}\delta_{\alpha' t'}\delta_{\beta' t}\,\delta(\vec{x}'-\vec{z})\delta(\vec{x}-\vec{z}')$$

$$= \delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{ss'}\delta_{tt'}\delta(\vec{z}-\vec{z}')\delta(\vec{x}-\vec{x}') \qquad (J3)$$

Now multiply (J3) by $\langle S's'_3|s\alpha\rangle\langle s'\beta|S''s''_3\rangle\langle T't'_3|t\alpha'\rangle\langle t'\beta'|T''t''_3\rangle$ and sum over $\alpha,\alpha',\beta,\beta',s,s',t$ and t'; this gives

$$\sum_{\substack{Ss_3\alpha\beta\\Tt_3ss'\\\alpha'\beta'tt'}} \int d^3q_1 d^3q_2 \langle S's'_3|s\alpha\rangle\langle s\alpha|Ss_3\rangle\langle T't'_3|t\alpha'\rangle\langle t\alpha'|Tt_3\rangle\langle S''s''_3|s'\beta\rangle \times$$

$$\times \langle s'\beta|Ss_3\rangle\langle T''t''_3|t'\beta'\rangle\langle t'\beta'|Tt_3\rangle F^{ST}_{q_1q_2}(x_o,\vec{x},\vec{z})F^{ST*}_{q_1q_2}(x_o,\vec{x}',\vec{z}')$$

$$+ \sum_{\substack{\alpha\beta ss'\\\alpha'\beta'tt'}} \delta_{\alpha s'}\delta_{\beta s}\delta_{\alpha' t'}\delta_{\beta' t}\,\delta(\vec{x}'-\vec{z})\delta(\vec{x}-\vec{z}')\langle S's'_3|s\alpha\rangle \times$$

$$\times \langle s'\beta|S''s''_3\rangle\langle T't'_3|t\alpha'\rangle\langle t'\beta'|T''t''_3\rangle$$

$$= \sum_{\substack{\alpha\beta ss'\\\alpha'\beta'tt'}} \delta_{\alpha\beta}\delta_{\alpha'\beta'}\delta_{ss'}\delta_{tt'}\delta(\vec{z}-\vec{z}')\delta(\vec{x}-\vec{x}')\langle S's'_3|s\alpha\rangle \times$$

$$\times \langle s'\beta|S''s_3''\rangle\langle T't_3'|t\alpha'\rangle\langle t'\beta'|T''t_3''\rangle$$

i.e.

$$\int d^3q_1 d^3q_2 \; F^{S'T'}_{q_1q_2}(x_o,\vec{x},\vec{z}) F^{S'T'*}_{q_1q_2}(x_o,\vec{x}',\vec{z}') \delta_{S''S'} \delta_{s_3''s_3'} \delta_{T''T'} \delta_{t_3''t_3'}$$

$$+ \sum_{ss'tt'} \langle S's_3'|ss'\rangle\langle s's|S''s_3''\rangle\langle T't_3'|tt'\rangle\langle t't|T''t_3''\rangle \delta(\vec{x}-z')\delta(\vec{x}'-\vec{z})$$

$$= \sum_{st\beta\beta'} \langle S's_3'|s\beta\rangle\langle s\beta|S''s_3''\rangle\langle T't_3'|t\beta'\rangle\langle t\beta'|T''t_3''\rangle \delta(\vec{x}-\vec{x}')\delta(\vec{z}-\vec{z}') \qquad \text{(J4)}$$

But

$$\sum_{\substack{ss'\\tt'}} \langle S's_3'|ss'\rangle\langle s's|S''s_3''\rangle\langle T't_3'|tt'\rangle\langle t't|T''t_3''\rangle$$

$$= \sum_{\substack{ss'\\tt'}} (-1)^{1-S''} (-1)^{1-T''} \langle S's_3'|ss'\rangle\langle ss'|S''s_3''\rangle\langle T't_3'|tt'\rangle\langle tt'|T''t_3''\rangle$$

$$= (-1)^{S''+T''} \; \delta_{S'S''} \delta_{s_3''s_3'} \delta_{T'T''} \delta_{t_3''t_3'}$$

Therefore (J4) becomes

$$\int d^3q_1 \; d^3q_2 \; F^{ST}_{q_1q_2}(x_o,\vec{x},\vec{z}) F^{ST*}_{q_1q_2}(x_o,\vec{x}',\vec{z}')$$

$$= \delta(\vec{x}-\vec{x}')\delta(\vec{z}-\vec{z}') - (-1)^{S+T}\delta(\vec{x}'-\vec{z})\delta(\vec{x}-\vec{z}') \qquad \text{(J5)}$$

In deriving (J5), the bound states have been ignored; but they can be incorporated trivially, by merely adding on a term on the lhs. In terms of the centre of momentum wavefunctions, this then reduces to

$$\int d^3q\ f^{ST}(\vec{r}) f^{ST*}(\vec{r}') + \sum_n f_n^{ST}(\vec{r}) f_n^{ST*}(\vec{r}')$$

$$= \frac{1}{(2\pi)^3} [\delta(\vec{r}-\vec{r}') - (-1)^{S+T}\delta(\vec{r}+\vec{r}')] \ . \qquad \text{(J6)}$$

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